SPACE-TIME CORRELATIONS OF VELOCITY AND PRESSURE AND THE ROLE OF CONVECTION FOR HOMOGENEOUS TURBULENCE IN THE UNIVERSAL RANGE

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ABSTRACT

Kolmogorov's principles provide a basis for the treatment of Eulerian space-time correlations for turbulence in the universal range by explicit separation of the kinematic effect of convection by the large eddies. With reference to unsheared homogeneous turbulence, the usual similarity forms are assumed for intrinsic velocity correlations in a local co-moving frame. The local-convection approximation, neglecting dispersion in this frame and hence relating space-time correlations to pure spatial correlations, is indicated to be a useful one in the inertial and viscous subranges. For an isotropic normal velocity distribution, the structure functions of fluctuating velocity are computed and found to be nearly space-time isotropic in the former subrange and exactly so in the latter. The wavenumber-frequency spectrum of energy in the mean rest frame, $E_4(k,\omega)$, however, in the regime of large ω/v_0k, where v_0 denotes rms velocity, is not given correctly by the local-convection approximation, but essentially involves dispersion. Taylor's hypothesis relating wavenumber (k_1) spectra in the mean rest frame to frequency spectra in a measurement

frame with velocity $-\bar{u}_o$ is found, in the local-convection approximation for the inertial subrange, to be exactly valid for all v_o/u_o provided a particular effective convection velocity different from u_o is assumed; for sufficiently large k_1L, where L is the scale of energy-containing eddies, the dispersive correction to this result is negligible. A plausible explicit form is proposed for the intrinsic energy spectrum in the co-moving frame, and consequent corrections to results of the local-convection approximation for space-time correlations and spectra are computed. This dispersive correction to correlations is appreciable at Reynolds numbers of typical grid-turbulence experiments. An extension of the basic separation of convection and the local-convection approximation to shear flow is suggested.

From kinematic and similarity arguments for the inertial subrange, inferences are also made concerning pressure spectra in a measurement frame with velocity $(-u_o,0,0)$ relative to the unsheared flow. A similarity form results for the wavenumber-frequency spectrum $\hat{P}(\bar{K},\omega)[\bar{K} = (k_1,k_3)]$ in the vicinity of the convective ridge $(|\omega - k_1u_o| \lesssim v_oK)$. For $v_o/u_o \ll 1$, the functional form is then obtained for the point frequency spectrum $\hat{P}(\omega)(\propto \omega^{-7/3})$; the convective contribution to the spectrum $\hat{Q}(\omega)$ of average pressure on a moving circular area of radius R_o for $\omega R_o/u_o \gg 1$ is smaller by a factor $0.8(\omega R_o/u_o)^{-3}$. On assumption of quasinormality of the velocity distribution and use of the non-dispersive approximation of space-time isotropy for the longitudinal correlation, the spectrum $\hat{P}(\bar{K},\omega)$ in the vicinity

of the convective ridge is determined explicitly, as well as the cross-spectral density and space-time correlation of pressure. From quasinormality, $\hat{P}(\bar{k},\omega)$ is estimated also in the low-wavenumber domain. In the limit $\omega R_o/v_o >> 1$ the ratio of the low-wavenumber to the mean-convective contribution to the moving-area spectrum $\hat{Q}(\omega)$ is $\sim(v_o/u_o)^{10/3}(\omega R_o/u_o)$.

1. INTRODUCTION

In previous descriptive and phenomenological treatments of turbulence, Eulerian space-time correlations of fluctuating velocity have received limited attention relative to that accorded spatial correlations at a fixed time. Yet time correlations have been extensively measured, are amenable to familiar kinematic and similarity arguments, and have essential importance in a number of applications.

Eulerian time correlations, i.e., in the general sense, correlations of quantities measured in a coordinate frame whose uniform motion is defined independently of the decorrelating fluid motions in question, are influenced by convection. Convection, where it has meaning, is a kinematic effect and can be treated trivially and independently of the turbulent dynamics. Moreover, when the large-scale convective effects are separated out, it becomes possible to invoke similarity considerations to treat the effect of the dispersive residual fluid motion.

Accordingly, some principal objectives of the present paper are to elucidate the role and consequences of convection, assess the validity and explore the consequences and limitations of the local-convection (non-dispersive) approximation relating space-time correlations to purely spatial correlations, and to propose and apply a dispersive generalization based on similarity in the inertial subrange.

These questions are pursued in the spirit of Kolmogorov's principles; no effort is made to deal with the explicit dynamics of turbulent flow. Further verification or modification may presumably be expected from currently developing treatments of turbulent dynamics. The type of flow treated is stationary, unsheared, homogeneous, and incompressible.* A generalized form of the non-dispersive approximation is suggested also for shear flow.

This investigation originated in a concern for the properties of fluctuating pressure in a turbulent boundary layer. Accordingly, a further objective is to consider, in the light of the work on velocity spectra, also pressure spectra in unsheared homogeneous turbulence, not only relative to a frame in which the mean velocity vanishes, but also a measurement frame moving at a velocity $-\bar{u}_o$, say. We thereby introduce a convection velocity as a kinematic effect unrelated to the dynamics; thus $\bar{u}_o$ plays a role similar to that of the local mean flow velocity in boundary-layer turbulence but without the complication of a shear flow.

As it was the motivating context, we review briefly how the problem of pressure fluctuations in a turbulent boundary layer impels consideration of space-time correlations

*The assumption that the turbulence is stationary as well as homogeneous and isotropic implies, strictly speaking, the presence of isotropic, statistically homogeneous energy sources. Alternatively, we may regard the turbulence as decaying, but with a characteristic time large compared to any time intervals considered.

of velocity.* Consider the wavenumber-frequency spectrum of pressure in the plane bounding a turbulent flow, $P(\overline{K},\omega)$ $[\overline{K} = (k_1,k_3)]$. If the turbulent eddies were of frozen shape, an eddy convected downstream at speed u would generate pressure fluctuations at a given frequency ω only via its component having streamwise wave number $k_1 = \omega/u$; hence if the convection speed at no depth in the flow exceeds the asymptotic flow speed U_∞, the spectrum $P(\overline{K},\omega)$ would contain only wavenumbers $K \geq \omega/U_\infty$ $[K = |\overline{K}|]$. In actuality, though the wavenumber spectrum peaks above ω/U_∞, it does not vanish at lower wave numbers but has a "tail" there on account of non-convective effects, i.e., distortion and decay of eddies.

In certain applications, however, some wavenumber range in this tail with $K \ll \omega/U_\infty$ is heavily weighted relative to the range $K > \omega/U_\infty$. This circumstance is true of the frequency spectrum of pressure averaged over a large area, e.g., over the face of a flush-mounted transducer of radius R_0 where $\omega R_0/U_\infty \gg \pi$, by virtue of spatial averaging of the short-wavelength components with $K \gtrsim \pi R_0^{-1}$.** It is true also of the frequency spectrum of pressure on an area or at a point shielded from the flow by a layer of material of thickness $L \gg U_\infty/\omega$ and of large lateral extent, at low Mach number, on account of acoustic attenuation of disturbances having wave length smaller than that of sound in the intervening layer. Hence, to achieve a theoretical

*For elaboration of this discussion see Chase (1965).

**The spectrum of average pressure due to wave numbers $K > \omega/U_\infty$ in this situation has area dependence as R_0^{-3}.

account of the effect of area averaging for a flush area and of acoustic averaging for a shielded area, we must know how the wavenumber spectrum of pressure declines with decreasing K in the non-convective tail where $K < \omega/U_\infty$.*

Furthermore, this tail is relevant to the scaling of boundary-layer pressure spectra with the flow parameters, which has not been unambiguously established through the whole domain of interest. Specifically, by the usual relation between the pressure and its fluctuating velocity sources [Eq.(4-1)], contributions to pressure are attenuated exponentially with source depth x_2 and parallel wave number K, i.e. as $\exp(-Kx_2)$. At relatively high frequencies, therefore, the convective contribution to pressure, having $K > \omega/U_\infty$ will derive mainly from velocity sources not far outside the viscous sublayer. The characteristic length involved by this contribution is thus likely to be the sublayer-thickness parameter ν/v_*, where v_* is the friction velocity.** On the other hand, the non-convective, low-wavenumber contribution will derive from sources at greater average depth, perhaps extending through a substantial fraction of the boundary layer. The characteristic length

*In the very low-wavenumber range $K \lesssim \omega/c$, where c is the sound velocity in the fluid, it becomes necessary to consider compressibility.

**This conclusion would be certain if all the coordinate spatial scales of fluctuating velocity components in the transition layer varied as wall distance x_2; if some scales are determined by the large eddies, however, they will be of the order of the displacement thickness instead of the wall distance. Recently, Bradshaw (1965) has contributed to a grasp of this scaling dichotomy.

involved by this contribution may thus be rather the displacement thickness δ_*. Therefore, the scaling of the resultant average-pressure spectrum depends at each frequency on the appropriately weighted magnitude of the wavenumber spectrum of pressure $P(\overline{K},\omega)$, and hence of fluctuating velocity, in the non-convective range $K<\omega/U_\infty$ relative to that in the convective range $K>\omega/U_\infty$.*

*The scaling of the two contributions could be nearly equivalent if both are independent of the respective length parameters; the frequency spectrum of point pressure then would vary as ω^{-1}.

2. ROLES OF CONVECTION AND DISPERSION AND THE LOCAL-CONVECTION APPROXIMATION FOR EULERIAN SPACE-TIME CORRELATIONS AND SPECTRA OF VELOCITY

Let the Eulerian space-time correlation function between a quantity $\alpha(\bar{x},t)$ measured at position $\bar{x}$ at time t and a quantity $\beta(\bar{x}+\bar{r},t+\tau)$ measured at $\bar{x}+\bar{r}$, $t+\tau$ in a given coordinate frame be represented, on assumption of a stationary homogeneous process, by

$$(2\text{-}1) \qquad \langle\alpha(\bar{x},t)\beta(\bar{x}+\bar{r},t+\tau)\rangle = W_{\alpha\beta}(\bar{r},\tau),$$

where <> denotes an ensemble average. Let $\hat{W}_{\alpha\beta}(\bar{r}',\tau)$ represent the correlation function for the same two measurement events where $\bar{r}'$ refers to the spatial separation in a frame moving relative to the first at the constant velocity $-\bar{u}$. The Galilean transformation between frames yields the familiar functional relation

$$(2\text{-}2) \qquad \hat{W}_{\alpha\beta}(\bar{r},\tau) = W_{\alpha\beta}(\bar{r}-\bar{u}\,\tau,\tau).$$

This rudimentary relation reflects the kinematic character of convection. In the wavenumber-frequency domain, the corresponding relation between the respective four-dimensional transforms becomes

$$(2\text{-}3) \qquad \hat{E}_{4\alpha\beta}(\bar{k},\omega) = E_{4\alpha\beta}(\bar{k},\omega-\bar{u}\cdot\bar{k}).$$

We turn explicitly to stationary homogeneous turbulence and consider the Eulerian correlation function (2-1) in the mean rest frame with α and β taken to be components of

fluid velocity, say v_i and v_j. Since we shall consider the universal equilibrium range, we prefer ordinarily to deal with the velocity structure function (multiplied by ½); we denote this two-point, two-component tensor by $\Psi_{ij}(\bar{r},\tau)$:

$$\Psi_{ij}(\bar{r},\tau) = \tfrac{1}{2} \langle [v_i(\bar{x} + \bar{r},\ t+\tau) - v_i(\bar{x},t)][v_j(\bar{x}+\bar{r},t+\tau) - v_j(\bar{x},t)] \rangle. \tag{2-4}$$

For i = j, this quantity is the decrease of the autocorrelation from unity; we shall call such quantities "decorrelations."

2.1 SEPARATION OF THE LARGE-SCALE CONVECTIVE EFFECT

We now restrict consideration to spatial separations $\bar{r}$ and time delays τ that correspond to the universal equilibrium range, namely

$$r \ll L,\ v_o |\tau| \ll L, \tag{2-5}$$

where v_o represents the rms magnitude of fluctuating velocity and L the characteristic scale of the large, energy-containing eddies. Together, the conditions (2-5) insure that the spatial separation of the two correlated space-time points in a frame convected with the local velocity ($\sim v_o$) associated with the large eddies (an average of the local velocity at the two points) is small compared to the size of the large eddies. Later we shall restrict consideration to Reynolds numbers high enough for existence of an inertial subrange where viscous effects are negligible.

It is an underlying assumption of the Kolmogorov (1941) theory of the universal range that the large eddies convect the small eddies without directly distorting them. A related, and probably consequent, assumption is that, viewed in the local frame in which the motion due to the large eddies vanishes, the small eddies are statistically independent of the large. These points have been emphasized by Kraichnan (1964) and were recognized, for example, by Heisenberg (1948), Von Weizsäcker (1948), and Silverman (1957).

We may thus consider a subset of realizations of the velocity field all corresponding to approximately the same local velocity $\overline{v}$ in a given space-time region of small size (defined by (2-5)) and take a statistical average of the two-point velocity product over this partial ensemble. By the stated assumptions, the resulting velocity correlation tensor, if expressed as a function of coordinates in the frame with velocity $\overline{v}$, will be independent of $\overline{v}$. We shall call the Eulerian correlation function referring to this frame, which is locally co-moving for any given convection velocity $\overline{v}$ due to the large eddies, the intrinsic correlation function, and designate the corresponding decorrelation tensor as $\tilde{\Psi}_{ij}(\overline{r},\tau)$ [cf. (2-4)]. For $\overline{r}=0$, $\tilde{\Psi}_{ij}(0,\tau)$ is the time correlation in a frame which for each realization may be taken as that in which the velocity at the initial time t vanishes; the Eulerian time correlation so defined is closely related to the Lagrangian time correlation and is equal to it in the limit of small τ

and perhaps for all τ. We may alternatively view the partial ensemble average at fixed $\overline{v}$ as a spatial average over a region of dimensions larger than r (the separation in the comoving frame) and larger than $v_o\tau$, but small compared to L.

In view of relation (2-2), for fixed $\overline{v}$ the decorrelation tensor in the mean rest frame is related to the intrinsic decorrelation by

$$\Psi_{ij}(\overline{r},\tau) = \tilde{\Psi}_{ij}(\overline{r}-\overline{v}\tau,\tau).$$

We let $P(\overline{v})d^3\overline{v}$ denote the probability that the fluid velocity in the mean rest frame lies in the three dimensional element $d^3\overline{v}$ of velocity space. Performing now the average over the local convection velocity $\overline{v}$ due to the large eddies, which is identified in the approximation in question with the total fluid velocity, we obtain the basic relation of the standard Eulerian space-time correlation (2-4) to the intrinsic decorrelation:

(2-6) $$\Psi_{ij}(\overline{r},\tau) = \int d^3\overline{v}P(\overline{v})\ \tilde{\Psi}_{ij}(\overline{r}-\overline{v}\tau,\tau).$$

Setting $\tau = 0$, we note the obvious relation

(2-6a) $$\Psi_{ij}(\overline{r},0) = \tilde{\Psi}_{ij}(\overline{r},0).$$

The pure spatial decorrelation $\Psi_{ij}(\overline{r},0)$, we recall, according to Kolmogorov's principles is an isotropic tensor. Likewise, by the related earlier assumption, the intrinsic decorrelation $\tilde{\Psi}_{ij}(\overline{r},\tau)$ is isotropic even for $\tau\neq 0$. It does

not generally follow, however, that $\Psi_{ij}(\bar{r},\tau)$ is isotropic for $\tau=0$, in spite of condition (2-5), since in (2-6) the probability density $P(\bar{v})$ referring to the large eddies may not be isotropic, i.e., it may depend on the direction of $\bar{v}$.*,**

Relation (2-6) separates out the purely kinematic, convective effect of the large-scale eddies from the instrinsic small-scale properties of the turbulence.*** The equivalent of relation (2-6) was given in the present context by Silverman (1957). An instance of use of a relation of the same sort where the function having the role of $\hat{\Psi}_{ij}(\bar{r},\tau)$ does not depend on the argument τ is provided, as Silverman noted, by the standard treatment of the scattering of radio waves by aggregates of discrete scatterers in random motion (e.g., Kerr (1951)).

While the separation (2-6) in the universal range is doubtlessly valid to some approximation, it is not entirely clear how far this approximation may be pushed without resort to Kolmogorov's principles in a questionably strong form.

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*We could also abandon the earlier assumption of homogeneity in the domain (2-5) and write a generalization of Eq. (2-6) where $P(\bar{v})$ would depend also on position and $\Psi_{ij}(\bar{r},\tau)$ would no longer be homogeneous unless $\tau = 0$.

**Since $\hat{\Psi}_{ij}(\bar{r},\tau)$ is symmetric in i and j, according to (2-6) so also is $\Psi_{ij}(\bar{r},\tau)$ even if $P(\bar{v})$ is anisotropic.

***This distinction between fluctuations in local convection velocity and distortion of eddies has been emphasized by Fisher and Davies (1963) in connection with intense shear turbulence.

In particular, it is not obvious that such a relation, with $\tilde{\Psi}_{ij}(\bar{r},\tau)$ taken independent of $\bar{v}$ and of all properties of the large energy-containing eddies, is valid beyond the local-convection approximation (to be considered in the following section) in which $\tilde{\Psi}_{ij}(\bar{r},\tau)$ is replaced by $\Psi_{ij}(\bar{r},0)$. This point is discussed further in the Appendix [see also Kraichnan (1964)]. Nevertheless, the presumption that the intrinsic decorrelation $\tilde{\Psi}_{ij}(\bar{r},\tau)$, for a suitably defined local velocity $\bar{v}$, has the independence in question has long received tentative acceptance [Von Weizsäcker (1948), Heisenberg (1948), Kraichnan (1959)] and is the simplest plausible explicit assumption; therefore, we accept it here.

We record for further reference the relations that follow from (2-6) between spectra in the proper and mean rest frames. First, we introduce the trace of the intrinsic decorrelation tensor:

(2-7) $\qquad \tilde{\Psi}(r,\tau)=\tilde{\Psi}_{ii}(\bar{r},\tau),$

where a sum is understood on repeated indices and $\tilde{\Psi}$, being an isotropic scalar, depends on $\bar{r}$ only via $r(\equiv|\bar{r}|)$. $\tilde{\Psi}$ is thus related to the trace $\tilde{W}(r,\tau)$ of the proper correlation tensor $\tilde{W}_{ij}(\bar{r},\tau)$ by

(2-8) $\qquad \tilde{\Psi}(r,\tau) = v_o^2 - \tilde{W}(r,\tau).$

A proper wavenumber spectrum of turbulent energy generalized to time delays $\tau \neq 0$ is defined in the usual way (e.g.Hinze (1959)) by

$$(2\text{-}9)\qquad \tilde{E}(k,\tau) = (2\pi)^{-2}k^2 \int d^3\bar{r}\, e^{-i\bar{k}\cdot\bar{r}}\, \tilde{W}(r,\tau)$$

$$= \pi^{-1}\int_0^\infty dr\, kr \sin kr\, \tilde{W}(r,\tau).$$

A similarly defined three-dimensional spectral correlation $\tilde{E}_{ij}(\bar{k},\tau)$, on account of isotropy in the proper frame, is related to $\tilde{E}(k,\tau)$ of (2-9) according to

$$(2\text{-}10)\qquad \tilde{E}_{ij}(\bar{k},\tau) = (2\pi)^{-3}\int d^3\bar{r}\, e^{-i\bar{k}\cdot\bar{r}}\, \tilde{W}_{ij}(\bar{r},\tau)$$

$$(2\text{-}11)\qquad = (4\pi)^{-1}(\delta_{ij}-k_ik_jk^{-2})k^{-2}\,\tilde{E}(k,\tau).$$

We may define the complete wavenumber-frequency spectrum of energy for the proper frame by

$$(2\text{-}12)\qquad \tilde{E}_4(k,\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\,\tilde{E}(k,\tau),$$

and likewise the spectral tensor $\tilde{E}_{4ij}(\bar{k},\omega)$ which is identically related to $\tilde{E}_{ij}(\bar{k},\tau)$. Eq. (2-11) yields

$$(2\text{-}13)\qquad \tilde{E}_{4ij}(\bar{k},\omega) = (4\pi)^{-1}(\delta_{ij}-k_ik_jk^{-2})k^{-2}\tilde{E}_4(k,\omega).$$

From the inverses of (2-9) and (2-12), the decorrelation trace is related to the wavenumber-frequency spectrum of energy of

$$(2\text{-}14)\qquad \tilde{\Psi}(r,\tau) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\omega\int d^3\bar{k}\,[1-e^{i(\bar{k}\cdot\bar{r}-\omega\tau)}]\,k^{-2}\,\tilde{E}_4(k,\omega).$$

It is sometimes convenient to express this as

$$(2\text{-}15)\qquad \tilde{\Psi}(r,\tau) = 4\int_0^\infty d\omega\int_0^\infty dk\,[(1-\cos\omega\tau)\sin kr/kr+(1-\sin kr/kr)]$$

$$\times\,\tilde{E}_4(k,\omega).$$

For the rest frame, analogously to (2-10), the three-dimensional spectral correlation is defined by*

(2-16) $$E_{ij}(\bar{k},\tau) = (2\pi)^{-3} \int d^3\bar{r}\, e^{-i\bar{k}\cdot\bar{r}}\, W_{ij}(\bar{r},\tau).$$

Substitution into (2-16) of Eq. (2-6) for W_{ij} in terms of $\tilde{W}_{ij}$, transformation of the integration vector from $\bar{r}$ to $\bar{r}-\bar{v}\tau$, and identification of a factor $\tilde{E}_{ij}(\bar{k},\tau)$ in the result yield

(2-17) $$E_{ij}(k,\tau) = (4\pi)^{-1}(\delta_{ij}-k_ik_jk^{-2})k^{-2}E(\bar{k},\tau),$$

where

(2-18) $$E(\bar{k},\tau) = \tilde{E}(k,\tau)M(\bar{k}\tau)$$

with

(2-19) $$M(\bar{k}\tau) = \int d^3\bar{v}\, P(\bar{v})\, e^{-i\bar{k}\cdot\bar{v}\tau}.$$

As in the proper frame, we may define a complete spectral tensor by

(2-20) $$E_{4ij}(\bar{k},\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau}E_{ij}(\bar{k},\tau).$$

Inserting (2-17) for $E_{ij}(\bar{k},\tau)$, we obtain the rest-frame analog of Eq. (2-13):

(2-21) $$E_{4ij}(\bar{k},\omega) = (4\pi)^{-1}(\delta_{ij}-k_ik_jk^{-2})k^{-2}E_4(\bar{k},\omega)$$

where

(2-22) $$E_4(\bar{k},\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}E(\bar{k},\tau).$$

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* Joint wavenumber frequency spectra of the kind defined here were introduced, for example, by Bass (1954).

From (2-18) we then find, in accord with (2-3),

$$(2\text{-}23) \qquad E_4(\bar{k},\omega) = \int d^3\bar{v}\; P(\bar{v})\; \tilde{E}_4(k,\omega-\bar{v}\cdot\bar{k}).$$

Eq. (2-23) [or (2-18)] is the spectral equivalent of the basic relation (2-6). By the inverse of (2-16) and (2-20),

$$(2\text{-}24) \qquad \Psi_{ij}(\bar{r},\tau)=(4\pi)^{-1}\int_{-\infty}^{\infty} d\omega \int d^3\bar{k}\,(\delta_{ij}-k_ik_jk^{-2})\,[1-e^{i(\bar{k}\cdot\bar{r}-\omega\tau)}]$$
$$\times k^{-2}E_4(\bar{k},\omega),$$

whence also the trace $\Psi(\bar{r},\tau)$ is related to the energy spectrum $E_4(\bar{k},\omega)$ as in the proper frame analog (2-14). The usual wavenumber spectrum of energy at fixed time, $E(k)$, we note, may be written as

$$(2\text{-}25) \qquad E(k)=\tilde{E}(k,0) = \int_{-\infty}^{\infty} d\omega\; \tilde{E}_4(k,\omega)$$
$$= E(\bar{k},0) = \int_{-\infty}^{\infty} d\omega E_4(\bar{k},\omega).$$

If the distribution $P(\bar{v})$ of the large-scale turbulent flow is anisotropic, $E_{4ij}(\bar{k},\omega)$, unlike $\tilde{E}_{4ij}(\bar{k},\omega)$, is not an isotropic tensor, but according to (2-21) it is the product of an isotropic tensor and an anisotropic scalar, $E_4(\bar{k},\omega)$; by (2-17) the same is true of $E_{ij}(\bar{k},\tau)$. It is useful to define also the frequency transform of the spatial correlation tensor,

$$(2\text{-}26) \qquad \Theta_{ij}(\bar{r},\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\; W_{ij}(\bar{r},\tau)$$
$$= (4\pi)^{-1} \int d^3\bar{k}\; e^{i\bar{k}\cdot\bar{r}}\; (\delta_{ij}-k_ik_jk^{-2})k^{-2}E_4(\bar{k},\omega).$$

Since $\tilde{\Psi}(r,\tau)$ is isotropic, we may define longitudinal and transverse decorrelation functions $\tilde{\Psi}_f(r,\tau)$ and $\tilde{\Psi}_g(r,\tau)$, referring to $\tilde{\Psi}_{ij}(\bar{r},\tau)$ for $i=j$ with the direction i respectively parallel and orthogonal to $\bar{r}$*. By definition,

$$\text{(2-27)} \qquad \tilde{\Psi}(\bar{r},\tau) = \tilde{\Psi}_f(r,\tau) + 2\,\tilde{\Psi}_{\rho}(r,\tau).$$

The continuity equation implies

$$\text{(2-28)} \qquad \tilde{\Psi}_g(r,\tau) = \tilde{\Psi}_f(r,\tau) + \tfrac{1}{2}r\,\partial\tilde{\Psi}_f(r,\tau)/\partial r.$$

In terms of $\tilde{\Psi}_f$ and $\tilde{\Psi}_g$, we have

$$\text{(2-29)} \qquad \tilde{\Psi}_{ij}(\bar{r},\tau) = r_i r_j r^{-2}[\tilde{\Psi}_f(r,\tau)-\tilde{\Psi}_g(r,\tau)] + \delta_{ij}\tilde{\Psi}_g(r,\tau),$$

and Eq. (2-6) becomes

$$\text{(2-30)} \qquad \Psi_{ij}(\bar{r},\tau) = \int d^3\bar{v}P(\bar{v})\;\{\rho_i\rho_j\rho^{-2}[\tilde{\Psi}_f(\rho,\tau)-\tilde{\Psi}_g(\rho,\tau)]+\delta_{ij}\tilde{\Psi}_g(\rho,\tau)\},$$

where $\bar{\rho} = \bar{r}-\bar{v}\tau$.

Now we assume approximate isotropy and define $P(\bar{v})v^2dv$ as the probability for a velocity magnitude between v and $v+dv$, whence

$$\text{(2-31)} \qquad P(\bar{v}) = (4\pi)^{-1}P(v).$$

Acceptance of (2-31) in (2-19) yields

$$\text{(2-32)} \qquad M(\bar{k}\tau) \equiv M(kv_o\tau) = \int_0^\infty dv\; v^2P(v)\;\sin kv\tau/kv\tau,$$

where the argument $kv_o\tau$ recognizes that the scale of v is the rms value v_o. We may rewrite relation (2-18) as

---

*Thus the normalized longitudinal correlation function $\tilde{f}(r,\tau)$ referring to the proper frame may be written $\tilde{f}(r,\tau) = 1-(3/v_o^2)\tilde{\Psi}_f(r,\tau)$; the function commonly denoted by $f(r)$ is $\tilde{f}(r,0)$.

$$(2\text{-}33) \qquad E(k,\tau) = \tilde{E}(k,\tau)M(kv_o\tau),$$

as given by Silverman (1957) (also see Heisenberg 1948 , Wandel and Kofoed-Hansen 1962). We may similarly write (2-23) as

$$(2\text{-}34) \qquad E_4(k,\omega) = \int d^3\overline{v}P(\overline{v})\ \tilde{E}_4(k,\omega-\overline{v}\cdot\overline{k})$$

$$= \tfrac{1}{2}\int_0^\infty dv\ v^2P(v)\int_{-1}^{1} d\mu\tilde{E}_4(k,\omega-kv\mu).$$

Finally, in this case of complete isotropy, the trace function $\Psi(r,\tau) = \Psi_{ii}(\overline{r},\tau)$ of the decorrelation tensor in the rest frame, together with the continuity equation, suffices to determine this tensor (and depends on $\overline{r}$ only via r), just as $\tilde{\Psi}(r,\tau)$ determines $\tilde{\Psi}_{ij}(\overline{r},\tau)$. Hence, the relation (2-6) becomes equivalent to the relation

$$(2\text{-}35) \qquad \Psi(r,\tau) = \int d^3\overline{v}P(\overline{v})\tilde{\Psi}(|\,\overline{r}-\overline{v}\tau\,|,\tau).$$

The one-point velocity distribution $P(\overline{v})$ is expected to be normal, on the basis, roughly, of the central limit theorem; within experimental error, this result has long been experimentally verified, as discussed by Batchelor (1956). A normal distribution would be expected for the components v_i of $\overline{v}$ even in the event of anisotropy where the principal axes of the ellipsoid of constant probability would be unequal. In the approximation of isotropy, the normal distribution is given by

(2-36) $$P(v) = 3^{3/2}\,(2/\pi)^{\frac{1}{2}}\,v_o^{-3}\exp(-3v^2/2v_o^2);$$

then the convection factor M of (2-32) and (2-33) becomes

(2-37) $$M(kv_o\tau) = \exp(-k^2v_o^2\tau^2/6).$$

2.2 LOCAL-CONVECTION APPROXIMATION FOR SPACE-TIME CORRELATIONS BASED ON SIMILARITY AND THE CHARACTER OF VELOCITY DISPERSION

Having delineated the relations between correlations and spectra in the proper and rest frames, we consider the intrinsic decorrelation $\tilde{\Psi}(r,\tau)$ on the basis of the usual dimensional arguments for the universal range. We also consider the deviation of $\tilde{\Psi}(\rho,\tau)$ from the zero-delay value $\tilde{\Psi}(\rho,0)$ for the range of $\rho \equiv |\bar{r}-\bar{v}\tau|$ that is significant in the integral (2-6) with r and τ restricted by (2-5). We thereby tentatively justify the local-convection approximation for relating the rest-frame space-time correlation to the purely spatial correlation.

Because of convection [Eq. (2-6)], the rest-frame decorrelation $\Psi(\bar{r},\tau)$ depends on the large eddies via the rms fluctuating velocity v_o even in the universal range.* By the

*Unless $P(\bar{v})$ is isotropic, $\Psi(\bar{r},\tau)$ depends on still other characteristics of the large eddies.

discussion at the beginning of Sec. 2.1, however, the intrinsic decorrelation is independent of the large eddies (Von Weizäcker 1948, Heisenberg 1948] see also Kraichnan 1964). Hence, by Kolmogorov's similarity principles, $\tilde{\Psi}(r,\tau)$ in the universal range (2-5)* must have the form (Chandrasekhar 1956)

$$(2\text{-}38) \qquad \tilde{\Psi}(r,\tau) = (\epsilon\nu)^{1/2}\, F(\xi_1,\xi_2),$$

where F is a universal function of the arguments

$$\xi_1 = \epsilon^{1/4}\, \nu^{-3/4} r,\ \xi_2 = \epsilon^{1/2} \nu^{-1/2} \tau$$

with ϵ the energy dissipation rate per unit mass and ν the kinematic viscosity.

We now restrict consideration to the inertial sub-range where, in addition to conditions (2-5), either the separation r or the rms wander in the proper frame during τ is large compared to the microscale. This condition is expressed by**

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*It may suffice here, since $\tilde{\Psi}(r,\tau)$ refers to the co-moving frame, to define the universal range by $r \ll L$ and $\epsilon^{1/2}|\tau|^{3/2} \ll L$ in place of (2-5); this condition on τ implies a pseudo-Lagrangian excursion small compared with L and is slightly weaker than $v_0|\tau| \ll L$, since it may be expressed roughly as $(v_0|\tau|)^{3/2} \ll L^{3/2}$. In any case, in one degree of another it is required that $v_0|\tau| \ll L$, since if $v_0|\tau| \sim L$, an initially co-moving frame after time τ no longer removes the influence of even the large, energy-containing eddies.

**I.e., if $\epsilon^{1/3} r^{-2/3} |\tau| \geq 1$, the rms Lagrangian separation of the fluid elements at the correlation points $(\bar{x},t)$, $(\bar{x}+r,\ t+\tau)$ is $\sim\epsilon|\tau|^{3/2}$; if $\epsilon^{1/3} r^{-2/3} |\tau| \leq 1$, this separation is rather $\sim r$ (see discussion below). Condition (2-39) thus insures that the effective separation is large relative to the microscale.

(2-39) $\quad (r^2 + \epsilon\,|\,\tau\,|^3)^{\frac{1}{2}} >> \ell_o$ or $(\xi_1{}^2 + |\,\xi_2\,|^3)^{\frac{1}{2}} >> 1,$

where

(2-40) $\quad \ell_o \equiv \nu^{3/4}\epsilon^{-\frac{1}{4}}.$

Later we shall consider separately the viscous range. In the inertial subrange, as well known, independence of ν implies the more restrictive form*

(2-42) $\quad \tilde{\Psi}(r,\tau) = (\epsilon r)^{2/3}F(z),$

where F is a universal function of

(2-43) $\quad z \equiv \epsilon^{1/3}r^{-2/3}\,|\,\tau\,|.$

Introducing $v_r \equiv (\epsilon r)^{1/3}$, which to within a constant is the velocity dispersion $\tilde{\Psi}(r,0)$ over a distance r at fixed time, we may write (2-42) also as

$$\tilde{\Psi}(r,\tau) = v_r^2\, F(v_r\tau/r).$$

Referring to the limiting cases $z = 0$ and $z \rightarrow \infty$, we have the standard forms implied by similarity arguments,

(2-44) $\quad \tilde{\Psi}(r,0) = A_o(\epsilon r)^{2/3},$

(2-45) $\quad \tilde{\Psi}(0,\tau) = B\epsilon\,|\,\tau\,|,$

whence

(2-46) $\quad F(0) = A_o, \quad \lim\limits_{z\rightarrow\infty}[z^{-1}F(z)] = B,$

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*On assumption of an isotropic convection-velocity distribution $P(\overline{v})$, we note, relation (2-6) and form (2-42) imply a rest-frame decorrelation of the form

(2-41) $\quad \Psi(r,\tau) = (\epsilon r)^{2/3}H(v_o\tau/r,z) \equiv (\epsilon r)^{2/3}\, H(v_o\tau/r, v_r\tau/r).$

where A_o and B are positive universal constants, presumably of the order of unity. Experimental results quoted by Corrsin (1963) for the constant N in the ordinary wavenumber spectrum

$$(2\text{-}47) \qquad E(k) = N\epsilon^{2/3}k^{-5/3}$$

yield $N \simeq 1.5$, whence, $A_o \simeq 3.6$; likewise a semi-empirical analysis by him suggests $B \simeq 4.7$.* These estimates, especially the latter, must be regarded as rough, but at any rate there is no suggestion of a gross disparity between the values of A_o and B in (2-44) and (2-45).

F(z) in (2-42) increases from the value A_o at $z = 0$ to become Bz at large z. It is reasonable to presume a smooth behavior for F(z) in between. On this presumption, since $B \sim A_o$, we may tentatively suppose that F(z) differs from A_o by only a small fraction where $z \ll 1$. This supposition can be confirmed only by suitable comparisons with experiment, but it is useful to give a brief heuristic discussion. $\tilde{\Psi}(\rho,\tau)$ differs from $\tilde{\Psi}(\rho,0)$, i.e. $F(z_\rho)$, with $z_\rho \equiv \epsilon^{1/3}\rho^{-2/3}|\tau|$, differs from A_o, on two conceptually distinguishable accounts. First, there is a velocity difference between the points separated by ρ whose mean squared magnitude is $v_\rho^2 \sim \tilde{\Psi}(\rho,0) = A_o(\epsilon\rho)^{2/3}$.

*Values $N = 1.77$ and $B = 17.5$ have recently been estimated theoretically by Kraichnan (1966).

This velocity acts as a residual differential convection velocity that cannot be removed by the motion of the proper frame and, during τ, produces an rms displacement $v_\rho \tau$; hence, the mean squared magnitude of the velocity dispersion between the two points occurring on this account during τ is $\sim\tilde{\Psi}(v_\rho|\tau|,0) \sim A_o(\epsilon v_\rho|\tau|)^{2/3} \sim A_o(\epsilon\rho)^{2/3} z_\rho^{2/3}$. Second, the velocity even at a fixed point in the proper frame does not remain constant, so that, even if $\rho \to 0$, some decorrelation occurs during time τ. The mean squared magnitude of this change is $\sim \tilde{\Psi}(0,\tau) \doteq B\epsilon|\tau| \doteq B(\epsilon\rho)^{2/3} z_\rho$. Summing the two contributions in quadrature, we obtain an estimate of the total velocity dispersion due to time delay as a fraction of that for zero delay:

$$(2\text{-}48) \qquad [\tilde{\Psi}(\rho,\tau) - \tilde{\Psi}(\rho,0)]/\tilde{\Psi}(\rho,0) \doteq [F(z_\rho) - F(0)]/F(0)$$

$$\sim z_\rho^{2/3} + (B/A_o) z_\rho .$$

This fraction, in fact, is small if $z_\rho \ll 1$. Actually, by an argument given in the Appendix [see also Sec.5], the contribution in (2-48) from spatial velocity dispersion more likely varies as z_ρ^2 than as $z_\rho^{2/3}$ for small z_ρ, and the contribution from velocity wander may not enter in quadrature but in such a way as to be of still higher order ($\propto z_\rho^3$), though at large z_ρ it must vary as z_ρ and predominate. In fact, in Sec. 3.1 on the basis of a specific conjectured intrinsic spectrum $\tilde{E}_4(k,\omega)$ and without reference to the heuristic distinction of spatial velocity dispersion and velocity wander, it is found that $[F(z_\rho) - F(0)]/F(0) \simeq 0.47(B/A_o)^2 z_\rho^2$, and apart from

the value of the numerical coefficient the same result holds for a broad class of possible $\tilde{E}_4(k,\omega)$.

For the moment we suppose $P(\overline{v})$ is isotropic; it then suffices to consider the trace function $\Psi(r,\tau)$. Inserting (2-42) in (2-35) we have*

$$(2\text{-}51) \qquad \Psi(r,\tau) = \epsilon^{2/3}\int d^3\overline{v}\ P(\overline{v})\,|\overline{r} - \overline{v}\tau|^{2/3}F(\epsilon^{1/3}|\tau|\,|\overline{r}-\overline{v}\tau|^{-2/3}).$$

In view of the behavior $F(z)\rightarrow Bz$ as $z\rightarrow\infty$, there is no singular behavior of the integrand at $\overline{r} - \overline{v}\tau = 0$. When the integrations over angles of $\overline{v}$ are performed, the spatial and convective displacement vectors $\overline{r}$ and $\overline{v}\tau$ roughly speaking, add in quadrature. With the final integration over $\overline{v}$, the characteristic order of magnitude of $|\overline{r}-\overline{v}\tau|$ in the integrand is simply

$$(2\text{-}52) \qquad R_c \equiv (r^2+v_o{}^2\tau^2)^{\frac{1}{2}}.$$

The argument z involved in the integrand is thus typically of tne

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*Form (2-51) contrasts with that assumed by Chandrasekhar (1956) in his theory of turbulence. By his contention

$$(2\text{-}49) \qquad \Psi'(r,\tau) = \epsilon^{2/3}r^{-1/3}\sigma(z), \quad z = \epsilon^{1/3}r^{-2/3}|\tau|,$$

where a prime denotes $\partial/\partial r$. But integration over r with neglect of the contribution from the viscous range then yields

$$(2\text{-}50) \qquad \Psi(r,\tau) = (\epsilon r)^{2/3}\,\Sigma(z) + \Psi(0,\tau)\ , \text{ where}$$

$$\Sigma(z) = (3/2)z\int_z^\infty dx\ x^{-2}\sigma(x)$$

[cf.(2-42)]. According to (2-50), the large eddies contribute to the decorrelation only additively and independently of r. If the large-scale motion is isotropic and $\Psi(0,\tau)$ depends on it only via v_o, for example, (2-50) has the form

$$\Psi(r,\tau) = (\epsilon r)^{2/3}\,\Sigma(z) + (\epsilon v_o|\tau|)^{2/3}\,T(\epsilon^{\frac{1}{2}}|\tau|^{\frac{1}{2}}/v_o).$$

The failure of form (2-50) to reflect properly the effect of convection by large eddies was emphasized by Kraichnan (1959), (1959a).

order of

$$z_c \equiv \epsilon^{1/3} R_c^{-2/3} |\tau|.$$

By use of the usual estimate $\epsilon \sim v_o^3/L$, we then have

$$z_c \sim (v_o|\tau|/R_c)(R_c/L)^{1/3}.$$

The first factor is less than unity by definition (2-52), and the second is small relative to unity by the restriction (2-5) to the universal range. Hence only arguments $z \ll 1$ enter significantly in (2-51), and for such arguments, by our previous discussion, the fractional error will be small if we replace F in (2-51) by $F(0)$ [$\equiv A_o$]. In accord with (2-6a) we identify $(\epsilon r)^{2/3} F(0)$ as $\Psi(r,0)$. We must insert R_c in place of r as the pertinent argument in condition (2-39) for use of the inertial-subrange form (2-42) in the integral (2-51); since $\epsilon |\tau|^3 \ll v_o^2 \tau^2$ by (2-5), the inertial subrange in the present context is then defined by

(2-53) $\qquad \ell_o \ll R_c \ll L.$

We thus obtain, as a valid approximation in the inertial subrange (2-53), an explicit expression for the space-time decorrelation $\Psi(r,\tau)$ in terms of the purely spatial decorrelation $\Psi(r,0)$:

(2-54) $\qquad \Psi(r,\tau) \doteq \int d^3\overline{v}\ P(\overline{v})\ \Psi(|\overline{r} - \overline{v}\tau|, 0).$

In other words, in the computation of $\Psi_{ij}(\overline{r},\tau)$ by Eq. (2-6), the trace $\tilde{\Psi}(r,\tau)$, which specifies $\tilde{\Psi}_{ij}(\overline{r},\tau)$ in that equation,

is approximated by

$$(2\text{-}55) \qquad \tilde{\Psi}(r,\tau) \simeq \tilde{\Psi}(r,0) = \Psi(r,0).$$

We refer to (2-54) as the local-convection, or non-dispersive, approximation. This relation has been compared with experiment by Favre (1965), who attributes its proposal to Kovásznay. We shall consider these experiments later. In Sec. 3.4 we shall derive explicitly in certain limits the dispersive correction to (2-54) given by (2-51) for the $F(\mathbf{z})$ derived from the specific conjectured form of $\tilde{E}_4(\mathbf{k},\omega)$ mentioned above. The dispersive correction to the local-convection approximation (2-54) for $\Psi(r,\tau)$ (or to the corresponding approximations for $\Psi_{ij}(\bar{r},\tau)$) is found, for the class of $\tilde{E}_4(k,\omega)$ considered, to be of relative order $\sim(B/A_o)^2 \times(\epsilon\tau/v_o{}^2)^{2/3} \sim(B/A_o)^2(v_o\tau/L)^{2/3}$ and, still more generally, vanishes in the defining limit $R_c/L \to 0$ of the universal range. We discuss in the next section limitations on the validity of the wavenumber-frequency spectrum corresponding to (2-54).

In the more general case of anisotropic $P(\bar{v})$, the trace $\Psi(\bar{r},\tau)$ does not suffice to determine $\Psi_{ij}(\bar{r},\tau)$. Exactly as at (2-42), however, for the inertial subrange we may write

$$(2\text{-}56) \qquad \tilde{\Psi}_i(r,\tau) = A_i(\epsilon r)^{2/3}\,\sigma_i(\mathbf{z}), \quad (i = 0,1,2)$$

where $\tilde{\Psi}_o \equiv \tilde{\Psi}$, $\tilde{\Psi}_1 \equiv \tilde{\Psi}_f$, $\tilde{\Psi}_2 \equiv \tilde{\Psi}_g$ and the equation for $i=0$ repeats (2-42) with $\sigma_0(z) = F(z)/A_o$. We may make $\sigma_o(0) = \sigma_1(0) = \sigma_2(0) = 1$ by taking $(3/11)A_o = (3/4)A_2 = A_1 \equiv A$ and defining A accordingly.

The $\Psi_{ij}(\overline{r},\tau)$ can be formed directly by inserting $\tilde{\Psi}_f$, $\tilde{\Psi}_g$ as given by (2-56) into (2-30). The earlier discussion of Eq. (2-51) applies substantially unchanged to these equations for $\Psi_{ij}(\overline{r},\tau)$ since $\sigma_1(z)$, $\sigma_2(z)$ are similar in character to $F(z)$*. Hence the local-convection approximation (2-54) may be generalized to the form required for anisotropic $P(\overline{v})$:

(2-59) $$\Psi_{ij}(\overline{r},\tau) \approx \int d^3\overline{v}\ P(\overline{v})\Psi_{ij}(\overline{r}-\overline{v}\tau,0).$$

This approximation is still equivalent to application of the approximation (2-55) in Eq. (2-6).

We turn to consider whether (2-54) [and (2-59)] should be valid also in the viscous range. In more general terms, our previous arguments may be summarized as follows. The effective displacement argument $|\overline{r}-\overline{v}\tau|$ in Eq.(2-6) for $\Psi(r,\tau)$ is characteristically $\sim R_c$. Hence approximation (2-54) will be valid if

(2-60) $$[\tilde{\Psi}(R_c,\tau)-\tilde{\Psi}(R_c,0)]/\tilde{\Psi}(R_c,0) \ll 1.$$

Considering two types of contribution, we estimate

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*More explicitly, the continuity equation (2-28) with (2-27) and (2-56) yields $\sigma_2(z) = \sigma_1(z) - 1/4\ z\sigma_1'(z)$, $\sigma_0(z) = \sigma_1(z)-(2/11)z\sigma_1'(z)$. For the coefficients $b_i \equiv \lim_{z\to\infty}[\sigma_1(z)/z]$, these imply $(9/11)b_1 = (12/11)b_2 = b_0$. From these results we find

(2-57) $\sigma_1(z) = (11/2)z^{11/2} \int_z^\infty dz'\ z'^{-13/2}\ \sigma_0(z')$.

(2-58) $\sigma_2(z) = (11/8)[\sigma_0(z)-(3/2)z^{11/2} \int_z^\infty dz'\ z'^{-13/2}\sigma(z')]$.

$$\tilde{\Psi}(r,\tau) - \tilde{\Psi}(r,0) \lesssim \tilde{\Psi}([\tilde{\Psi}(r,0)]^{\frac{1}{2}}\,|\tau|,0) + \tilde{\Psi}(0,\tau).$$

Hence the condition for (2-54) becomes

(2-61) $\quad [\tilde{\Psi}([\tilde{\Psi}(R_c,0)]^{\frac{1}{2}}\,|\tau|,0) + \tilde{\Psi}(0,\tau)]/\,\tilde{\Psi}(R_c,0) \ll 1.$

In the viscous range we have the standard result (Hinze (1959))

(2-62) $\quad \tilde{\Psi}(\rho,0) \simeq 5(\epsilon/30\nu)\rho^2 \quad \text{for } \rho \lesssim \ell_o,$

corresponding to $F(\xi_1,0) = 5(1/30)\xi_1^2$ in Eq. (2-38) for $\xi_1 \lesssim 1$. Similarly, for $\xi_2 \lesssim 1$ we must have $\tilde{\Psi}(0,\tau) \propto \tau^2$, i.e. $F(0,\xi_2) = 5c_2\xi_2^2$, where c_2 is a constant; thus

(2-63) $\quad \tilde{\Psi}(0,\tau) \approx 5c_2(\epsilon\nu)^{\frac{1}{2}}(\epsilon/\nu)\tau^2 \quad \text{for } \tau \lesssim (\nu/\epsilon)^{\frac{1}{2}}.$

This form in the similar case of the Lagrangian correlation is implied also by the work of Uberoi and Corrsin (1953), summarized by Hinze (1959), p. 321. Identifying the two cases, we would infer from this work a value for c_2 given by

(2-64) $\quad c_2 \sim (15)^{-3/2}(12.75\alpha^{-1} + 4.46\alpha),$

where α denotes Heisenberg's dimensionless spectral-transfer constant, presumed of the order of unity. By insertion of (2-62)-(2-63), condition (2-61) may be written

(2-65) $\quad (1/6)\xi_2^2 + 30c_2\xi_2^2/\xi_{1c}^2 \ll 1.$

where $\xi_{1c} = \epsilon^{\frac{1}{4}}\nu^{-3/4}R_c$. In the viscous range defined by

(2-66) $\quad (r^2 + \epsilon^{1/2}\nu^{1/2}\tau^2)^{1/2} \lesssim \ell_o$ or $(\xi_1^2 + \xi_2^2)^{1/2} \lesssim 1,$

we have $(1/6)\xi_2^2 \ll 1$ in (2-66). Condition (2-65) then reduces to

(2-67) $\quad 30c_2 \, [(\epsilon\nu)^{\frac{1}{2}}/v_o^{\,2}][1 + (r/v_o\tau)^2]^{-1} \ll 1.$

But

(2-68) $\quad v_o^{\,2}/(\epsilon\nu)^{\frac{1}{2}} = (3/\sqrt{15})Re_\lambda,$

where Re_λ is the usual turbulence Reynolds number based on transverse dissipation scale, and $Re_\lambda \gg 1$ for the assumed existence of the universal equilibrium range. According to (2-64), we have $c_2 \sim \frac{1}{4}$. Hence (2-67) is satisfied, and the local-convection approximation (2-54) or (2-59) is expected to hold in the viscous range (2-66), as well as in the inertial subrange (2-53).

Conceivably, (2-54) is less valid in the connecting region between inertial and viscous ranges. Apart from this possibility, the local-convection approximation is expected to be valid throughout the universal range where $R_c \ll L$.

2.3 SPACE-TIME CORRELATIONS BY THE LOCAL-CONVECTION APPROXIMATION FOR AN ISOTROPIC NORMAL VELOCITY DISTRIBUTION; SPACE-TIME ISOTROPY; COMPARISON WITH EXPERIMENT

We assume now a flow having an isotropic and normal distribution of total fluctuating velocity $P(\overline{v})$ given by (2-31) and (2-36). In such case we may define longitudinal and transverse decorrelation functions for the mean rest frame, $\Psi_f(r,\tau)$ and $\Psi_g(r,\tau)$. The relations (2-27) - (2-29) must then hold as well for the rest-frame functions (all tildes omitted). Eq.(2-6) for i=j with i respectively parallel and orthogonal to $\overline{r}$ then yields integrals for Ψ_f, Ψ_g in terms of $\tilde{\Psi}_f, \tilde{\Psi}_g$. Using spherical coordinates in $\overline{v}$-space ($d^3\overline{v} = dv d\mu d\emptyset$) with $\overline{r}$ as polar axis, and representing (Ψ, Ψ_f, Ψ_g) by (Ψ_o, Ψ_1, Ψ_2) for convenience (whence $\Psi_o = \Psi_1 + 2\Psi_2$), we have*

$$(2\text{-}69) \qquad \Psi_i(r,\tau) = (1/2)\int_o^\infty dv\, v^2 P(v) \int_{-1}^{1} d\mu\, \Psi_i(\mu),$$

with

$$(2\text{-}70) \qquad \Psi_o(\mu) \equiv \tilde{\Psi}_f(\rho,\tau) + 2\tilde{\Psi}_g(\rho,\tau),$$

$$(2\text{-}71) \qquad \Psi_1(\mu) \equiv (\overline{r}-v\tau\mu)^2 \rho^{-2}[\tilde{\Psi}_f(\rho,\tau) - \tilde{\Psi}_g(\rho,\tau)] + \tilde{\Psi}_g(\rho,\tau),$$

$$(2\text{-}72) \qquad \Psi_2(\mu) \equiv (1/2)(v\tau)^2(1-\mu^2)\, \rho^{-2}[\tilde{\Psi}_f(\rho,\tau) - \tilde{\Psi}_g(\rho,\tau)] + \tilde{\Psi}_g(\rho,\tau),$$

$$\rho^2 = r^2 + (v\tau)^2 - 2rv\tau\mu.$$

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*As a check, it is verified directly that Ψ_f, Ψ_g satisfy the continuity equation (2-28) provided $\tilde{\Psi}_f, \tilde{\Psi}_g$ do.

We now utilize the local-convection approximation and consider first the inertial subrange. We then have the $\tilde{\Psi}_i(r,\tau)$ given by Eq.(2-56) with the $\sigma_i(z)$ replaced by unity. Inserting (2-36) in (2-69) and integrating over μ, we obtain

$$\Psi_i(r,\tau) = 3(6/\pi)^{\frac{1}{2}}A(\epsilon r)^{2/3} \int_0^\infty dx\, x^2 \exp(-3x^2/2)L_i(xv_o\tau/r), \tag{2-73}$$

where

$$L_o(\beta) = (11/16)\beta^{-1}(s_+^{8/3} - s_-^{8/3}),$$

$$s_+ = |1+\beta|, \quad s_- = |1-\beta|,$$

and L_1 and L_2 are given by similar but more complicated expressions.

We readily find the limiting forms of (2-73):

$$\Psi_i(r,\tau) \rightarrow A_i\ (\epsilon r)^{2/3}(1+c_{io}\beta_o^2) \text{ for } \beta_o \ll 1, \tag{2-74}$$

$$\Psi_i(r,\tau) \rightarrow S_i(\epsilon v_o\tau)^{2/3}(1+c_{i\infty}\beta_o^{-2}) \text{ for } \beta_o \gg 1, \tag{2-75}$$

where

$$\beta_o \equiv v_o|\tau|/r,$$

$$A_2 = (4/3)A\ , \ A_o = (11/3)A,\ A_1 = A, \tag{2-76}$$

$$S_1 = S_2 = (2/3)^{1/3}(22/9)\pi^{-\frac{1}{2}}\Gamma(11/6)A,\ S_o = 3S_1,$$

$$c_{10} = 11/27,\ c_{20} = 11/108,\ c_{oo} = 5/27,$$

$$c_{1\infty} = 1/5,\ c_{2\infty} = 2/5,\ c_{o\infty} = 1/3.$$

The characteristic dependence $\Psi_i(0,\tau) \rightarrow S_i(\epsilon v_o\tau)^{2/3}$ for $r=0$

is well known (Landau and Lifshitz 1959, Eq. (32.2)).

We could also derive Ψ_1 and Ψ_2 from the simpler Ψ_o by the continuity equation. Thus, writing Eq. (2-73) functionally as

$$\text{(2-77)} \qquad \Psi_i(r,\tau) = A_i(\epsilon r)^{2/3} G_i(v_o\tau/r),$$

(whence $G_i(0)=1$), from (2-27) and (2-28) we obtain

$$\text{(2-78)} \qquad G_o(\beta_o) = G_1(\beta_o) - (3/11)\beta_o \partial G_1(\beta_o)/\partial\beta_o$$

(cf.(2-58)). Eq. (2-78), together with the asymptotic form $G_i(\beta_o) \to (S_i/A_i)\beta_o^{2/3}$ as $\beta_o \to \infty$, yields

$$\text{(2-79)} \qquad G_1(\beta_o) = (11/3)\beta_o^{11/3} \int_{\beta_o}^{\infty} d\beta\beta^{-14/3} G_o(\beta).$$

Since the displacements r and $v_o\tau$ must add nearly in quadrature for appropriate relative scaling, we define functions $H_i(\beta_o)$ by writing

$$\text{(2-80)} \qquad \Psi_i(r,\tau) = A_i(\epsilon R_i)^{2/3}[1-H_i(\beta_o)],$$

where

$$\text{(2-81)} \qquad R_i^2 = r^2 + s_i^2 (v_o\tau)^2, \quad s_i^2 = (S_i/A_i)^3.$$

By definition (2-81) of the scaling factors s_i^2 and the results (2-74), (2-75), the functions $H_i(\beta_o)$ so defined are such that

$$\text{(2-82)} \qquad H_i(0) = H_i(\infty) = 0,$$

i.e. $\Psi_i(r,\tau) = A_i(\epsilon R_i)^{2/3}$ if either $r = 0$ or $\tau = 0$. From the numerical values (2-76), we find

$$s_1^2 = 1.455,\ s_2^2 = 0.613,\ s_o^2 = 0.795. \tag{2-83}$$

The integrals (2-73) have been evaluated numerically, and the resulting functions $H_i(\beta_o)$ of (2-80) are shown in Figure 1 as functions of $\mathrm{Tan}^{-1}\beta_o$ [$\equiv \mathrm{Tan}^{-1}(v_o|\tau|/r)$].*

We shall refer to a function that depends on r and τ only via $r^2 + s^2(v_o\tau)^2$ for some constant s as a space-time isotropic function. Figure 1 shows that the $\Psi_i(r,\tau)$ are all closely approximated by the space-time isotropic functions defined by neglecting the H_i in (2-80), i.e.,

$$\Psi_i(r,\tau) \approx A_i(\epsilon R_i)^{2/3}. \tag{2-84}$$

The corresponding maximum fractional errors are 1.3%, 3.5% and 2.4% for the longitudinal, transverse, and trace functions ($i = 1,2,0$), respectively.

We turn momentarily from the inertial to the viscous subrange, applying once more the local-convection approximation

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* By expressing $\Psi(r,\tau)$ by the inverse of (2-9) and $E(k,\tau)$ by (2-33) and (2-37), with $\tilde{E}(k,\tau)$ replaced by $E(k) = N\epsilon^{2/3}k^{-5/3}$ in the present approximation, we obtain

$$\Psi(r,\tau) = 2N\epsilon^{2/3} \int_o^\infty dk\ k^{-5/3}[1-(\sin kr/kr)\exp(-v_o{}^2\tau^2k^2/6)],$$

whence, according to Bateman (1954), p.74, (24), $\Psi(r,\tau)$ can be expressed also in terms of Kummer's confluent hypergeometric series.

(2-54). In this instance we have, as at (2-62)

(2-85) $\Psi_1(\rho) = a_\nu\rho^2$, $\Psi_2(\rho) = 2a_\nu\rho^2$, $\Psi_o(\rho) = 5a_\nu\rho^2$,

where $a_\nu = \epsilon/30\nu$. Eq. (2-69), with P(v) given by (2-36), now yields

(2-86) $\Psi_i(r,\tau) = A_{oi}(\epsilon/30\nu)\, R_{oi}^2$

with

(2-87) $R_{oi}^2 = r^2 + s_{oi}^2(v_o\tau)^2$,

$s_{01}^2 = 5/3$, $s_{02}^2 = 5/6$, $s_{oo}^2 = 1$, [cf.(2-83)]

$A_{01} = 1$, $A_{02} = 2$, $A_{00} = 5$.

Hence, in the viscous range the decorrelations $\Psi_i(r,\tau)$ are exactly space-time isotropic.

A space-time isotropic correlation function was proposed previously by Lilley and Hodgson (1960), App.D.

For comparison with experiment, we consider also space-time decorrelations, say $\hat{\Psi}_{ij}(\bar{r},\tau)$, measured in a frame having a fixed velocity $-\bar{u}_o = (-u_o,0,0)$ relative to the mean rest frame. By the kinematic relation (2-2), we have $\hat{\Psi}_{ij}(\bar{r},\tau) = \Psi_{ij}(\bar{r} - \bar{u}_o\tau,\tau)$. In the present approximation $\Psi_{ij}(\bar{r}',\tau)$ is isotropic and given in terms of $\Psi_f(r,\tau)$, $\Psi_g(r,\tau)$ as in Eq. (2-29). In particular, the decorrelation for the streamwise component

is expressed by

$$\hat{\Psi}_{11}(\overline{r},\tau) = \Psi_f(r',\tau) + R^2(r_1'^2 + R^2)^{-1}[\Psi_g(r',\tau)-\Psi_f(r',\tau)], \tag{2-88}$$

where $\overline{r}' = (r_1-u_o\tau, r_2, r_3)$ and $R^2 = r_2^2 + r_3^2$. In the inertial subrange Ψ_f and Ψ_g are given by (2-80), but since a relative error ~1% is negligible, we use the space-time isotropic approximation (2-84) for Ψ_f and the consequent Ψ_g [Eq. (2-28)], obtaining

$$\hat{\Psi}_{11}(\overline{r},\tau) = A(\epsilon\hat{R}_1)^{2/3}(1+\tfrac{1}{3} R^2/\hat{R}_1^{\,2}), \tag{2-89}$$

where $\hat{R}_1^2 = (r_1-u_o\tau)^2 + R^2 + s_1^{\,2}v_o^{\,2}\tau^2$. The correlation at a point in the mean-rest frame after a delay τ according to (2-89) is the same as in the laboratory frame after a delay $\simeq (s_1v_o/u_o)\tau \sim (v_o/u_o)\tau$; this result was given by Lilley and Hodgson (1960), App.D, and noted to be in broad agreement with the results of Favre (1965) for both grid and wall turbulence.

The velocity component referred to here is parallel to $\overline{u}_o$ by definition, and in the measurement of present concern $\overline{r}$ is also nominally parallel to $\overline{u}_o$. We therefore suppose the angle ϕ [$= \sin^{-1}(R/r)$] between $\overline{r}$ and $\overline{u}_o$ is small, as well as $v_o/u_o \ll 1$, whence (2-89) may be written approximately as

$$\hat{\Psi}_{11}(\overline{r},\tau) \simeq A\epsilon^{2/3}[\Delta^2 + r^2\phi^2 + s_1^2(v_o/u_o)^2(r+\Delta)^2]^{1/3} \tag{2-90}$$

$$\times \{1 + \tfrac{1}{3} r^2\phi^2/[\Delta^2 + r^2\phi^2 + s_1^2(v_o/u_o)^2(r+\Delta)^2]\},$$

where

$$\Delta = u_o(\tau-\tau_m) \text{ and } \tau_m = r_1/u_o \approx r/u_o .$$

Thus τ_m is the time delay at which $\partial \hat{\Psi}_{11}/\partial r_1 = 0$ for given $\overline{r}$, and Δ is the actual time delay in spatial units relative to τ_m. Since $\phi \ll 1$ and $s_1 v_o/u_o \ll 1$, the time correlation at fixed separation is determined mainly by $A\epsilon^{2/3}$, but the correlation envelope curve $\partial\hat{\Psi}_{11}/\partial\tau = 0$ for varying separation is determined mainly by the product of $A\epsilon^{2/3}$ and the function of ϕ and v_o/u_o given by setting $\Delta = 0$ in (2-90). The rate of decrease of this correlation envelope is sensitive to the misalignment ϕ, being significantly increased by ϕ unless $\phi \ll v_o/u_o$.

We compare results computed from (2-90) with measurements in grid-produced turbulence by Favre, Gaviglio, & Dumas (1953),(1954), summarized by Favre (1965). Favre performed a similar comparison based directly on the nondispersive relation (2-64) (with $\overline{r} \to \overline{r} - \overline{u}_o\tau$) and employing the measured $\Psi_f(r',0)$, $\Psi_g(r',0)$. The present comparison entails the further approximation of Ψ_f and Ψ_g by the inertial-subrange forms but eliminates certain unassessed approximations made by Favre. The parameter A is now regarded as adjustable on the basis not only of the measured correlations for $\tau = 0$ but the whole series of measured space-time correlations $\Psi_{11}(\overline{r},\tau)$ and their envelope. Use of the inertial-subrange form, however, implies that the computed normalized space-time

correlation must be expected to depart substantially from the measured values at least where these are less than ~0.6. In the specific experiments referred to, in fact, the Reynolds number $Re_\lambda \approx 44$, scarcely sufficient for existence of an inertial subrange at all, and small enough also that the dispersive departure from the local-convection approximation may be appreciable, as discussed in Sec. 3.4 below.

For the geometrical and flow parameters given for the Favre experiments, we estimate from other experiments on decay of turbulence downstream of a grid, notably Batchelor and Townsend (1948), that the turbulence intensity at the pertinent locations was $v_0/u_0 \simeq 0.027$.* Accepting this value, assuming no misalignment ($\phi = 0$), and adjusting A to give a visually good fit to the form of the measured time correlations at various fixed separations r, with more emphasis on agreement where the correlation is relatively high, we arrive at $A \simeq 0.55$ and the series of curves in Figure 2 having the solid-line envelope.** The breadth of the peaks for various separations is fairly satisfactory relative to the experimental results shown, but the envelope of the peaks decreases too slowly. This fact is attributed mainly to

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*The dissipation rate ϵ is then computed by Eq. (2-72).

**At separation r = 0, for which the experimental curves given in Figs. 1 and 5 of Favre (1965) fail to coincide, we accept the latter.

neglect of dispersion and to experimental misalignment.* If, for example, the latter were $\phi = 1.5^{\circ}$ for each separation, with $v_o/u_o = 0.027$ still and $A = 0.55$, the curves would be those represented ——-—— ; these agree rather well with the measured ones and their envelope. If we suppose instead $\phi = 0$ but arbitrarily adjust the intensity as well as A to yield a fit, for $A = 0.41$ and $v_o/u_o = 0.042$ we obtain almost identical results. In the measurements, the nominal transverse separation was varied to maximize the correlation; hence the error due to misalignment was presumably small. It appears quite plausible that, if comparison were made with experiments at sufficiently high Reynolds number, the results yielded by (2-89) for space-time velocity correlations on the basis of the local-convection approximation would prove in satisfactory agreement with measurement, and provide a useful determination of the Kolmogorov constant N. Further comment will be made in Sec. 3.4, where a dispersive correction is introduced.

*We may consider also the effect of possible anisotropy of the large eddies. Grant and Nisbet (1957) obtained experimental indication that the streamwise fluctuating component in grid turbulence tends to be the largest. Making the extreme assumption of purely streamwise fluctuation in place of (2-31) we find, for $\bar{r}$ parallel to $\bar{u}_o$, that $\Psi_f(0,\tau)/\Psi_f(r,0) = 0.44(v_o\tau/r)^{2/3}$ in place of $1.13\,(v_o\tau/r)^{2/3}$. Hence such anisotropy would be expected to reduce rather than increase the rate of decline of the space-time correlation envelope. This anisotropy has no effect on the ratio ρ_c/ρ_s mentioned below.

The measured streamwise space-time correlations ($R=0$) may be characterized also by the ratio of minor to major axis for the streamwise isocorrelation curves, which depends somewhat on the correlation value (Favre, Gaviglio, and Dumas 1953). More generally, we define ρ_s as one half the ratio of r_1-intercept ($\tau=0$) of a streamwise isocorrelation curve to the value of $u_o\tau$ where $r_1=u_o\tau$ on this curve; for an ellipse, ρ_s is nearly the ratio of axes. Transverse isocorrelation curves ($r_1=0$) were also measured in the experiments and may be characterized mainly by the ratio ρ_t of the R-intercept ($\tau=0$) to the $u_o\tau$-intercept ($R=0$). Finally, compensated isocorrelation curves ($r_1=u_o\tau$) were measured and may be characterized by the ratio ρ_c of the R-intercept ($r_1=0$) to the r_1-intercept ($R=0$).

For the viscous range, which applies only where the normalized correlation is very near unity, Bass (1954) derived for homogeneous turbulence the general relations $\rho_s=(a/b)^{1/2}/2$, $\rho_t\simeq 1/\sqrt{2}$, $\rho_c=\sqrt{2}\rho_s$, where a/b is a small undetermined quantity. Eqs. (2-86) and (2-88) of the local-convection approximation for the viscous range conform to these results and imply $a/b = (5/3)(v_o/u_o)^2$, whence $\rho_s\simeq 0.65\ v_o/u_o$. For the inertial subrange, on the other hand, Eq.(2-89) yields $\rho_s=\frac{1}{2}s_1 v_o/u_o\simeq 0.61 v_o/u_o$, $\rho_t=\frac{3}{4}(1+s_1{}^2 v_o{}^2/u_o{}^2)^{1/2}\simeq 0.65$, $\rho_c=(\frac{3}{4})^{3/2}s_1 v_o/u_o\simeq 1.30\rho_s$, which we may compare with experiment. The measured values of ρ_s, without dispersion, require somewhat too large v_o/u_o, as already discussed. The measured ρ_t varies in the vicinity 0.4 to 0.5, a value appreciably

smaller than the computed 0.65. Most strikingly, the measured ρ_c/ρ_s varies in the vicinity of 4 to 6, much larger than the computed 1.3. Introduction of dispersion, anisotropy of the large eddies, or streamwise inhomogeneity appears unsuitable to account for this discrepancy. An obvious, but unlikely, way to account for it, if the measured value is not grossly in error, is to suppose the spatial velocity correlations at fixed time are anisotropic and the streamwise component correlated over a much larger distance in the transverse than the streamwise direction.

2.4 VELOCITY SPECTRA IN THE LOCAL-CONVECTION APPROXIMATION; IMPORTANCE OF VELOCITY DISPERSION

We consider the implications of the local-convection approximation (2-55) for the rest-frame velocity spectra defined in Sec. 2.1. The corresponding approximation for the partial transform $\tilde{E}(k,\tau)$ is

(2-91) $$\tilde{E}(k,\tau) \cong \tilde{E}(k,0) \doteq E(k),$$

where $E(k) \doteq E(k,0)$. By definition (2-12), then, we have

(2-92) $$\tilde{E}_4(k,\omega) \cong E(k)\delta(\omega),$$

whence (2-23) becomes

(2-93) $$E_4(\bar{k},\omega) \cong E(k)\int d^3\bar{v}P(\bar{v})\delta(\omega - \bar{k}\cdot\bar{v}),$$

This equation states that in the approximation of local convection, a fluctuating velocity component with given frequency ω and wave number $\bar{k}$ can be generated only by virtue of a local convection velocity $\bar{v}$ of the "frozen" large eddies such that the projection of $\bar{v}$ in the direction of $\bar{k}$ generates a disturbance at frequency ω by motion of the fixed wave pattern. The restriction (2-53) of (2-54) to the inertial subrange, by the rest-frame analog of (2-15), implies the reciprocal restriction

(2-93a) $$L^{-1} << (k^2 + \omega^2/v_o^2)^{1/2} << \ell_o^{-1}$$

on (2-93) and (2-95) below.

In the case of isotropic $P(\bar{v})$, Eq. (2-93) is equivalent to the following relation derived from Eq. (2-34):

(2-94) $$E_4(k,\omega) \cong (1/2)k^{-1}E(k)\int_{\omega/k}^{\infty} dv\,vP(v).$$

With a normal distribution [Eq. (2-36)] this becomes

$$(2\text{-}95) \qquad E_4(k,\omega) \simeq 3^{1/2}(2\pi)^{-1/2}(v_o k)^{-1}E(k)\exp[-(3/2)(\omega/v_o k)^2]$$

Hence, if the normal distribution represents the true situation sufficiently well, for large $\omega/v_o k$ the local-convection approximation yields an exceedingly small energy density $E_4(k,\omega)$ on account of the minute probability of a local convection velocity $>\omega/k$. Because of the deformation and decay of eddies, however, we must suppose that $E_4(k,\omega)$ in reality is much larger; i.e., however good an approximation the local-convection approximation may be for $\psi_{ij}(\overline{r}, \tau)$ in the universal range, it is not adequate to yield a good approximation to such a demanding detail as the frequency-wave-number spectrum at large $\omega/v_o k$.

An example of the sensitivity of $E_4(\overline{k},\omega)$ in this regime is provided by comparing the spectrum corresponding exactly to the local-convection approximation in the inertial subrange with that corresponding to the space-time isotropic approximation for the longitudinal decorrelation $\psi_f(r,\tau)$ [Eq. (2-84)]; the latter sufficed to give $\psi_f(r,\tau)$ within 1.3% of the value given by the former for all values of $v_o\tau/r$ (Fig.1). Corresponding to (2-56) with $\sigma_o = 1$, we have the $E(k)$ of (2-47) with $N = (110/27)[\Gamma(1/3)]^{-1}A$. Then (2-33), (2-37) and (2-95) yield

(2-96) $E(k,\tau) = N\epsilon^{2/3}k^{-5/3}\exp(-k^2 v_0^2 \tau^2/6)$,

$$E_4(k,\omega) = (3/2\pi)^{1/2}N\epsilon^{2/3}v_0^{-1}k^{-8/3}\exp[-(3/2)(\omega/v_0 k)^2].$$

The space-time isotropic approximation (2-84) for $\psi_f(r,\tau)$, on the other hand, by a standard relation (Hinze 1959, Eq. (3-70) extended to $\tau \neq 0$) yields

(2-97) $E(k,\tau) = 2^{-11/6}[\Gamma(17/6)]^{-1}N\epsilon^{2/3}k^{-5/3}(s_1 v_0 \tau k)^{17/6}$

$$\times K_{17/6}(s_1 v_0 \tau k),$$

$$E_4(k,\omega) = \pi^{-1/2}(14/11)[\Gamma(7/3)/\Gamma(11/6)]N\epsilon^{2/3}(s_1 v_0)^{-1}$$

$$\times k^{-8/3}[1 + (\omega/s_1 v_0 k)^2]^{-10/3},$$

where s_1 was given at (2-83) and K_ν denotes the usual exponentially decreasing modified Bessel function.* For large $\omega/v_0 k$ the approximation (2-97) for $E_4(k,\omega)$ thus does not retain the rapid rate of decrease given by the exact local-convection form (2-96).

2.5 TAYLOR'S HYPOTHESIS, EFFECTIVE CONVECTION VELOCITIES, AND CROSS-SPECTRA IN THE LOCAL-CONVECTION APPROXIMATION

The present methods afford a simple quantitative treatment of departures from Taylor's hypothesis for the inertial subrange in homogeneous turbulence, a topic previously

*If the approximation of space-time isotropy (2-84) is applied to the trace $\psi(r,\tau)$ instead of $\psi_f(r,\tau)$, the approximation to (2-96) is less close than (2-97), with $E(k,\tau)$ and $E_4(k,\omega)$ instead containing factors $K_{11/6}(s_0 v_0 \tau k)$ and $[1 + (\omega/s_0 v_0 k)^2]^{-7/3}$

treated otherwise by Lin (1953), Uberoi and Corrsin (1953). We define the hypothesis here in terms of the measured and the hypothetical spectra which it is commonly invoked to relate. The hypothesis approximates the one-dimensional wavenumber (k_1) spectrum of a turbulence quantity in the mean rest frame by its frequency spectrum, multiplied by u_o at a fixed point in the measurement frame, the latter frame having velocity $-\bar{u}_o$ relative to the former, for a frequency $\omega = u_o k_1$. We consider first, for simplicity, the spectra corresponding to the trace of the velocity correlation, $W(\bar{r},\tau)$, rather than to the longitudinal or transverse components individually. The first spectrum of convern is thus the one-dimensional energy spectrum in the rest frame (multiplied here by $\frac{1}{2}$), given by

$$(2\text{-}98)\qquad E^{(1)}(k_1) = (2\pi)^{-1}\int_{-\infty}^{\infty} dr_1 e^{-ik_1 r_1} W(\bar{r}_1,0)$$

where $\bar{r}_1 = (r_1,0,0)$; the second is the measurement-frame analog of the trace of Eq. (2-26) with $\bar{r} = 0$,

$$(2\text{-}99)\qquad \hat{\theta}(\omega) \equiv \hat{\theta}(0,\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\hat{W}(0,\tau).$$

where $\hat{W}(\bar{r},\tau)$ refers to the frame with velocity $-\bar{u}_o = (-\bar{u}_o,0,0)$. We wish to consider the departure from unity of the "Taylor ratio"

$$(2\text{-}100)\qquad T \equiv u_o\theta(u_o k_1)/E^{(1)}(k_1).$$

Eq. (2-98) can be straightforwardly transformed to become, as given by Hinze (1959), Eqs. (3-72), (3-73),

$$E^{(1)}(k_1) = \int_{k_1}^{\infty} dk k^{-1} E(k) . \tag{2-101}$$

In Eq. (3-19) we may use the kinematic relation

$$\hat{W}(\bar{r},\tau) = W(\bar{r} - \bar{u}_o\tau,\tau). \tag{2-102}$$

By use of the equivalent of (2-14) written for the mean rest frame and of (2-23), Eq. (2-99) may be written

$$\hat{\theta}(\omega) = (2\pi)^{-1}\int d^3\bar{v}P(\bar{v})\int d^3\bar{k}k^{-2}E_4(k,\omega - \bar{k}\cdot(\bar{u}_o + \bar{v})) . \tag{2-103}$$

Here we shall find the Taylor ratio T in the local-convection approximation (2-92) and consider the dispersive effect in Sec. 3.3. From (2-103) and (2-92) we find

$$\hat{\theta}(\omega) = \int d^3\bar{v}P(v)|\bar{u}_o + \bar{v}|^{-1}\int_{\omega/|\bar{u}_o + \bar{v}|}^{\infty} dk k^{-1}E(k), \tag{2-104}$$

whence

$$u_o\hat{\theta}(u_o k_1) = \int d^3vP(v)I(v/u_o,\lambda)\int_{k_1 I(v/u_o,\lambda)}^{\infty} dk k^{-1}E(k), \tag{2-105}$$

where

$$I(v/u_o,\lambda) = [1 + 2(v/u_o)\lambda + (v/u_o)^2]^{-1/2}, \tag{2-106}$$

$\lambda = \nabla\cdot\overline{u}_o/vu_o$, and $\int d^3\overline{v}$ may be regarded as expressed in the spherical form $\int_o^\infty dvv^2\int_o^{2\pi}d\phi\int_{-1}^{1}d\lambda$. In this approximation T is given as the ratio of (2-105) to (2-101). From (2-105) and (2-106), we note, if the azimuthal integral $\int_o^{2\pi} d\phi P(\overline{v})$ is an even function of λ, then T differs from unity only by terms of order $(v_o/u_o)^2$; this will be so, in particular, if $P(\overline{v})$ is isotropic.

We now limit consideration to a power-law wave-number spectrum:

$$E(k) = c_n k^{-(n+1)} \tag{2-107}$$

In actual fact we are concerned with the inertial subrange case $n = 2/3$. The k-integrations in (2-101) and (2-105) may now be performed to yield a ratio

$$T = \int d^3\overline{v}P(\overline{v})(|\overline{u}_o + \overline{v}|/u_o)^n = \int d^3\overline{v}P(\overline{v})[I(v/u_o,\lambda)]^{-n}. \tag{2-108}$$

Form (2-107) implies $\psi(\overline{r},0)\propto r^n$, and from this, proceeding from the original forms (2-98) and (2-99) by use of (2-54) or recasting (2-108), we can express T in the alternative form

$$T = \psi(-\overline{u}_o\tau,\tau)/\psi(-\overline{u}_o\tau,0); \tag{2-109}$$

the right side, under the present assumptions, is properly independent of τ. For $n = 2/3$ and an isotropic normal $P(\overline{v})$ recalling (2-77) and (2-80) (for $i = 0$), we may write

$$(2\text{-}110) \qquad T = G_o(v_o/u_o) = [1 + s_o^2(v_o/u_o)^2]^{1/3}[1 - H_o(v_o/u_o)].$$

for $v_o/u_o \ll 1$, Eq. (2-74) then yields explicitly

$$(2\text{-}111) \qquad T \simeq 1 + (5/27)(v_o/u_o)^2.$$

More generally, for arbitrary v_o/u_o we can compute the correction factor T explicitly from (2-110) by use of the value of s_o [Eq. (2-83)] and the graph of the function $H_o(\beta_o)$ given in Figure 1.

We can proceed with the spectra corresponding to individual component correlations $W_{ij}(r,\tau)$ similarly to the above for the trace $W(\bar{r},\tau)$. Thus, we let $E_{ij}^{(1)}(k_1)$ be defined by (2-98) with $W(r_1,0)$ replaced by $W_{ij}(r_1,0)$, $\hat{\theta}_{ij}(\omega)$ by (2-99) with $\hat{W}(0,\tau)$ replaced by $\hat{W}_{ij}(0,\tau)$, and, analogously to (2-100), define

$$(2\text{-}112) \quad T_{ij} = u_o\hat{\theta}_{ij}(u_o k_1)/E_{ij}^{(1)}(k_1).$$

In particular, as at (2-109), in the local-convection approximation with a power-law spectrum (2-107), we find

$$(2\text{-}113) \qquad T_{ij} = \psi_{ij}(-\bar{u}_o\tau,\tau)/\psi_{ij}(-\bar{u}_o\tau,0),$$

which once more is properly independent of τ. For the inertial subrange and an isotropic normal $P(\bar{v})$, as at (2-110), we then have as Taylor's ratio for the longitudinal ($i = 1$) and transverse ($i = 2$) velocity components

$$(2\text{-}114) \qquad T_{ii} = G_i(v_o/u_o) = [1 + s_i^2(v_o/u_o)^2]^{1/3}[1 - H_i(v_o/u_o)]$$

(no sum on i), where again the s_i are given at (2-83) and the $H_i(v_o/u_o)$ for arbitrary v/u_o in Figure 1. For $v/u_o << 1$, by (2-74), as at (2-111), these correction factors are given by

$$(2\text{-}115) \qquad T_{11} = 1 + (11/27)(v_o/u_o)^2, \; T_{22} = 1 + (11/108)(v_o/u_o)^2$$

It is possible to re-interpret the computed correction factors. Suppose we define new ratios T', T'_{ij} obtained by replacing u_o by some other velocities u', u'_{ij}, respectively, in the definitions (2-100) and (2-112). From the power-law form (2-107) we readily find $T' = (u_o/u')^n T$, $T'_{ij} = (u_o/u_{ij})^n T_{ij}$. Since the T and T_{ij} proved to be independent of k_1, we can make $T' = T'_{ij} = 1$ by choosing $u' = u_o T^{1/n}$, $u_{ij} = u_o T_{ij}^{1/n}$. From (2-114), for example, we see that we can provide an exact Taylor's hypothesis by employing effective convection velocities

$$(2\text{-}116) \qquad u_{ii} = (u_o^2 + s_i^2 v_o^2)^{1/2}[1 - H_i(v_o/u_o)]^{3/2}$$

to compute $E_{ii}^{(1)}(k_1) = u_{ii}\theta_{ii}(u_{ii}k_1)$. This velocity differs for longitudinal and transverse components. For $v_o/u_o << 1$, by (2-115) we have

$$(2\text{-}117) \qquad u_{11} = u_o[1 + (11/18)v_o^2/u_o^2], \; u_{22} = u_o[1 + (11/72)v_o^2/u_o^2].$$

As for the dispersive correction to the local-convection approximation applied here, in Sec. 3.3 it is shown, for the $\widetilde{E}_4(k,\omega)$ considered, that dispersion contributes to Taylor's ratio an added term of the order $\sim(\epsilon u_o^{-3} k_1^{-1})^{2/3}$ $\sim(v_o/u_o)^2 (k_1 L)^{-2/3}$ and, more generally, the added term vanishes independently of v_o/u_o in the subject limit.

We note here a result for a measurement frame quantity generalizing $\hat{\theta}(\omega)$ of Eq. (2-99), namely the frequency transform $\hat{\theta}_{ij}(\bar{r},\omega)$ of the spatial velocity correlation (cross-spectral density). Analogous to the mean-rest-frame quantity $\theta_{ij}(\bar{r},\omega)$ of Eq. (2-26), $\hat{\theta}_{ij}$ is given by

$$(2\text{-}118) \qquad \hat{\theta}_{ij}(\bar{r},\omega) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\tau e^{i\omega\tau}\, W_{ij}(\bar{r} - \bar{u}_o\tau,\tau)$$

or similarly by the second form of (2-26) with $\omega \to \omega - \bar{u}_o \cdot \bar{k}$ in E_4. A result will be given only with neglect of dispersion and with the further approximation of space-time isotropy for the longitudinal decorrelation $\psi_f(r,\tau)$. The streamwise element $\hat{\theta}_{11}(\bar{r},\omega)$, where $\bar{u}_o = (u_o,0,0)$, is then found from (2-118) and (2-89), or from (2-97) and the alternate form of (2-118), to be given by

(2-119)

$$\hat{\theta}_{11}(\bar{r},\omega) = 2^{-11/6}[\Gamma(17/6)]^{-1} N\epsilon^{2/3} u_e^{-1} (\omega/u_e)^{-11/6} \exp(i\omega r_1/u_c)$$
$$\times r_s^{-1/6}[(\omega r_s/u_e) K_{5/6}(\omega r_s/u_e) - \tfrac{1}{2}(\omega R/u_e)^2 K_{1/6}(\omega r_s/u_e)],$$

where

$$R^2 = r_2^2 + r_3^2 \ , \ r_s^2 = \nu_e^2 r_1^2 + R^2,$$

$$u_e^2 = u_o^2 + s_1^2 v_o^2, \ \nu_e = s_1 v_o/u_e, \tag{2-120}$$

$$u_c = u_e^2/u_o = u_o(1 + s_1^2 v_o^2/u_o^2).$$

Eq. (2-119) holds for all v_o/u_o and reduces to the rest-frame quantity $\theta_{11}(\bar{r},\omega)$ for $v_o = 0$.* For $v_o/u_o \ll 1$, the contours of constant $|\hat{\theta}_{11}(\bar{r},\omega)|$ are elongated along the longitudinal axis relative to the transverse by a factor $\approx u_o/v_o$. The effective convection velocity u_c defined by writing the phase of $\hat{\theta}_{11}$ as $\omega r_1/u_c$ thus proves to differ from u_o as given by (2-120). We recall from the discussion of Sec. 2.4 the limitation that when $\omega r_s/u_c$ is large (and hence $|\hat{\theta}_{11}|$ small) the result (2-119) of the space-time isotropic approximation will conform neither to the true $\hat{\theta}_{11}$ including dispersion nor to the exact local-convection approximation.

At this point we may usefully discuss the relations among variously defined convection velocities. Even in the present unsheared homogeneous flow, these differ from the relative rest-frame velocity u_o and from one another, though, when $v_o/u_o \ll 1$, only by terms of the order of v_o^2/u_o^2. In

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*We note the result of the exact local-convection approximation for $\bar{r} = 0$: $\theta_{11}(0,\omega) = (2/3)^{1/3}\pi^{-1/2}\Gamma(5/6)N\epsilon^{2/3}v_o^{-1}(\omega/v_o)^{-5/3}$. This agrees with (2-119) for this case to within a constant, as it must on dimensional grounds.

unsheared flow, if the measurement frame coincides with the laboratory frame, the intensity v_o/u_o is small enough that such differences are unmeasurable. In some types of intense sheared turbulence, or in an envisioned experiment where the measurement frame has a velocity intermediate between that of the laboratory frame and the mean rest frame*, on the other hand, these differences may be appreciable. In any event it is illuminating to examine these differences in the present context of a simple unsheared turbulence with fixed u_o.

A convection velocity is defined with reference to some property of a correlation function or its transform in the measurement frame, and this velocity in general differs according to the property in question and also according to the quantity the correlation refers to, e.g. streamwise component of fluctuating velocity (along $\bar{u}_o$). We define convection velocities here by four such properties and refer for definiteness to this streamwise component; the definitions are summarized here in Table 1:

*Turbulence measurements in a moving frame have recently been performed in a different connection by Uzkan (1965).

No.	Defining Condition	Convection Velocity Defined
1	$\partial\hat{W}_{11}(\overline{r},\tau)/\partial r_1 = 0$ $(\hat{W}_{11} = \max)$	r_1/τ
2	$\partial\hat{W}_{11}(\overline{r},\tau)/\partial\tau = 0$ $(\hat{W}_{11} = \max)$	r_1/τ
3	$\hat{\theta}_{11}(\overline{r},\omega) = \exp(i\omega r_1/u_c)\lvert\hat{\theta}_{11}(\overline{r},\omega)\rvert$	u_c
4	$E_{11}^{(1)}(k_1) = u_{11}\hat{\theta}_{11}(u_{11}k_1)$	u_{11}

Table 1. Various Definitions of Convection Velocity

In general, the convection velocities so defined might not be constant but dependent on the pertinent variables. On a standard plot of the space-time correlation $\hat{W}_{11}(\overline{r},\tau)$ vs. time delay with the time-correlation curve for each streamwise separation r_1 displaced along the time axis by r_1/u_o, condition 1 corresponds to the envelope curve, whereas condition 2 corresponds to the curve joining the maxima of the individual curves. These definitions have been discussed by Wills (1964) in the more general context of shear flow, along with other definitions which, in the present case, coalesce with these. Condition 3 defines a convection velocity based on the phase of a cross-spectral density, as done for example, by Corcos (1964). Condition 4 [recall Eqs. (2-98) and (2-99)] defines a convection velocity based on Taylor's hypothesis relating spectra in frequency and wavenumber.

We give results for the convection velocities of Table 1 in the local-convection approximation, neglecting

dispersion, for the inertial subrange, and assume once more an isotropic normal distribution of fluctuating velocity. We also compare the results given by the space-time isotropic approximation to $\psi_f(r',\tau)$, corresponding to neglect of H_1 in Eq. (2-80).*

Since $\hat{W}_{11}(\bar{r},\tau) = W_{11}(\bar{r} - \bar{u}_o\tau,\tau)$ exactly, the convection velocity of condition 1 is just u_o if $W_{11}(\bar{r}',\tau)$ has a maximum for all τ and all r_2, r_3 at $r_1' = 0$. Examination based on the results (2-80) [and (2-74), (2-75)] for $\psi_f(r',\tau)$ and $\psi_g(r'\tau)$ shows that in fact $\psi_{11}(\bar{r}', \tau)$ (obtained from (2-30)), as well as ψ_f and ψ_g individually, have minima at $r_1' = 0$. This result is given also by the space-time isotropic approximation to $\psi_f(r',\tau)$. Hence the convection velocity of condition 1 is just u_o.

With regard to the convection velocity of condition 2, we find by use of Eq. (2-75) and (2-74), respectively, that this is given in two opposite limits by

$$\text{(2-121)} \quad u_o[1 + (5/3)v_o^2/u_o^2] \text{ for } (v_o/u_o)^2 + (R/v_o\tau)^2 \ll 1,$$

$$u_o[1 + (11/9)v_o^2/u_o^2] \text{ for } (v_o/u_o)^2 + (R/v_o\tau)^2 \gg 1,$$

apart from terms of higher order, where $R^2 = r_2^2 + r_3^2$. In the space-time isotropic approximation, on the other hand,

*We do not neglect H_2 to find $\psi_g(r',\tau)$ but accept the result implied by the approximation to ψ_f.

from (2-84) we find this velocity to be given independently of r_2, r_3, and τ by $u_o(1 + s_1^2\, v_o^2/u_o^2) \simeq u_o(1 + 1.455 v_o^2/u_o^2)$.

The convection velocity u_c defined by condition 3 has not been determined in the exact local-convection approximation. In the space-time isotropic approximation, however, it was given by (2-120). The latter is the same velocity that was given by condition 2 in this approximation. More generally, in any approximation in which, for some i, j and $\bar{r}$,

$$\hat{\theta}_{ij}(\bar{r},\omega) = A_{ij}(\bar{r},\omega)\ \exp(i\omega r_1/u_{cij}) \quad (A_{ij}\ \text{real}) \tag{2-122}$$

with the phase convection velocity u_{cij} independent of ω, and in addition $A_{ij}(\bar{r}_1,-\omega) = A_{ij}(\bar{r},\omega)$, it follows from the general relation

$$\partial\hat{W}_{ij}(\bar{r},\tau)/\partial\tau = -i\int d\omega e^{-i\omega\tau}\omega\hat{\theta}_{ij}(\bar{r},\omega)$$

that

$$[\partial\hat{W}_{ij}(r,\tau)/\partial\tau]_{r_1 = u_{cij}\tau} = 0,$$

i.e. in such case the convection velocities defined by conditions 2 and 3 are the same.

The convection velocity u_{1i} defined by condition 4 was given at Eq. (2-116) or, for $v_o/u_o \ll 1$, Eq. (2-117). In the space-time isotropic approximation, the result is given by omitting H_1 in (2-116); for $v_o/u_o \ll 1$, this becomes

$u_{11} = u_o(1 + \frac{1}{2} s_1^2 v_o^2/u_o^2) \simeq u_o(1 + 0.728 v_o^2/u_o^2)$. The convection velocity defined by Taylor's relation (condition 4) depends, we note, on the rest-frame space-time decorrelation $\psi_{11}(r',\tau)$ in a range of the space-to-time separation ratio $r_1'/v_o\tau[= (r_1-v_o\tau)/v_o\tau]$ opposite to that which determines the velocity defined by condition 2. In the former instance, by (2-113), the relevant ratio is $|r_1'|/v_o\tau = u_o/v_o$, whereas in the latter, by (2-121), it is $r_1'/v_o\tau \sim v_o/u_o$.

We note that in the exact local-convection approximation the convection velocities defined by conditions 1 and 4 are constant (i.e., independent of r_2, r_3, τ, and of k_1, respectively, while in the further approximation of space-time isotropy the velocities defined by all four are constant. The results are summarized, for $v_o/u_o \ll 1$ and $\left(r_2^2 + r_3^2\right)^{1/2} \ll v_o\tau$, in Table 2, which gives the values of a_o where the respective velocities are written as $u_o(1 + a_o v_o^2/u_o^2)$.

Defining Condition (Table 1)	Approximation Used	
	Exact local convection	Space-time isotropic longitudinal component
1	0	0
2	5/3 ($v_o/u_o \ll 1$, $R/v_o\tau \ll 1$)	$s_1^2 = 1.455$
3		$s_1^2 = 1.455$
4	11/18 ($v_o/u_o \ll 1$)	$s_1^2/2 = 0.728$ ($v_o/u_o \ll 1$)

Table 2. Coefficients a_o for various convection velocities $u_o(1 + a_o v_o^2/u_o^2)$ referring to the streamwise velocity component.

3. RESULTS WITH VELOCITY DISPERSION

We wish to consider more explicitly the departure from the local-convection approximation due to the spatial and temporal dispersion of velocity in the local proper frame. This step permits us to consider the imporaant spectral regime $\omega/v_0 k \gg 1$ where dispersion is the determining effect, and to derive corrections to quantities where the dominant effect is convection. We assume, as discussed in Sec. 2.1 and the Appendix, that Eq. (2-6) with an intrinsic decorrelation having the similarity form (2-42) remains competent to yield these corrections.

3.1 MODEL INTRINSIC VELOCITY SPECTRUM INCORPORATING DISPERSION

The required form (2-42) for $\tilde{\psi}(r,\tau)$ in the inertial subrange implies that in a corresponding domain the intrinsic wavenumber frequency spectrum $\tilde{E}_4(k,\omega)$ has the functional form

(3-1) $$\tilde{E}_4(k,\omega) = \epsilon^{1/3} k^{-7/3} G(\Omega)\,, \quad \Omega \equiv \epsilon^{-1/3} k^{-2/3} |\omega|\,.$$

The inertial subrange for $\tilde{\psi}(r,\tau)$ is defined by

(3-2) $$L \gg (r^2 + \epsilon|\tau|^3)^{1/2} \gg \ell_0$$

[see condition (2-39) and footnote 1, preceding page]; consideration of Eq. (2-15) indicates that the conjugate subrange of (3-1) for $\tilde{E}_4(k,\omega)$ is

(3-3) $$L^{-1} \ll (k^2 + \epsilon^{-1}|\omega|^3)^{1/2} \ll \ell_0^{-1}\,.$$

The form (3-1) insures that $\tilde{\psi}(r,\tau)$ has the requisite limiting forms (2-46), provided the zeroth and first moments of $G(\Omega)$ converge.

We introduce for $G(\Omega)$ in (3-1) a conjectural explicit form that meets the obvious requirements on this function and has vertain other plausible properties (see Appendix); specifically, we limit consideration first to the following class of functions indexed by m:

$$G_m(\Omega) = a_m \Omega^{-1/2} (1+\gamma_m{}^2\Omega^3)^{-m} \tag{3-4}$$

($m \geq 1$), where a_m and γ_m are constants. This form, with nonvanishing γ_m, correctly provides a finite spatial scale such that motion correlated over a distance $\sim k^{-1}$ is associated mainly with frequencies smaller than the similarity estimate $\sim \epsilon^{1/3} k^{2/3}$. Some results to be obtained are independent of the specific form of (3-4), apart from values of numerical coefficients, and are really consequences of similarity together with presumed asymptotic properties. The type of function (3-4) for integral m has been chosen to lead to tractable integrations.

In illustrating the effects of dispersion we give results on assumption that m = 1 in (3-4), corresponding to $\tilde{E}_4(k,\omega)$ independent of k in the limit $\Omega \to \infty$, and also on assumption that m = 3, corresponding to $\tilde{E}_4(k,\omega)$ varying as k^4 in this limit.* The latter assumption is favored by the following argument. The restriction (3-3) on the domain of

*Quantities linearly related to $\tilde{E}_4(k,\omega)$ can be obtained for m=3 from those for m=1 simply by differentiation.

the inertial-subrange form (3-1) suffices even when $k \lesssim L^{-1}$, at least in the case where even the large-scale turbulence is homogeneous and isotropic so that Eq. (2-15) provides a valid basis for (3-3) at low k. (For $k \lesssim L^{-1}$, condition (3-3) is equivalent to $1 \ll (\omega L/v_o)^{3/2} \ll L/\ell_o$.) Now, in order to have $E(k) \propto k^4$ for $k \ll L^{-1}$, as required in this case, by reference to (2-25) we must also have $\tilde{E}_4(k,\omega) \propto k^4$ for $k \ll L^{-1}$; presuming that this requirement on E(k) cannot be relaxed with regard to contributions from $\omega \gg v_o/L$, we must have $\tilde{E}_4(k,\omega) \propto k^4$ for sufficiently small k even in the domain of (3-1). This result implies $G(\Omega) \propto \Omega^{-19/2}$ as $\Omega \to \infty$. The requirement $\tilde{E}_4(k,\omega) \propto k^4$ as $k \to 0$ is likewise implied in the usual way, and hence $G(\Omega) \propto \Omega^{-19/2}$ as $\Omega \to \infty$ in the domain (3-3), if it is supposed that the intrinsic velocity spectral tensor $\tilde{E}_{4ij}(k,\omega)$ of Eq. (2-13) should be analytic at $k=0$. This limiting dependence, in the case of the assumed form (3-4), implies $m=3$.

From (3-4) via (2-15) we are able to find the limiting forms of the resulting function F(z) in the decorrelation (2-42):*

(3-5) $\quad F(z) \to A_o(1+\alpha_m z^2) \quad$ as $z \to 0$

(3-6) $\quad F(z) \to Bz(1+\epsilon_m z^{-3/2})$ as $z \to \infty$;

A_o and B are regarded as given independently of m, and in the chosen cases $m=1$ and $m=3$ we obtain for the coefficients α_m, ϵ_m

*See Appendix, Eqs. (A-19) and (A-22). In this and other calculations below, we have performed various quadratures with the aid of Bateman (1954).

$$(3\text{-}7) \qquad \alpha_1 = (8\sqrt{3}/5\pi)(B/A_o)^2 \simeq 0.881(B/A_o)^2$$

$$\alpha_3 = [7(55)/(27)^2]\alpha_1 \simeq 0.529\alpha_1$$

$$\epsilon_1 = (2\pi)^{-1/2}c^{3/2}(B/A_o)^{-3/2}$$

$$\epsilon_3 = (8/3)(27/55)^{3/2}\epsilon_1 \simeq 0.916\epsilon_1$$

and for the constants a_m and γ_m in (3-4)

$$a_1 = \pi^{-2}c^{3/2}A_o(A_o/B)^{1/2}, \quad \gamma_1^{\,2} = c^3(A_o/B)^3,$$

$$a_3 = (8/3)(27/55)^{3/2}a_1, \quad \gamma_3^{\,2} = (27/55)^3\gamma_1^{\,2},$$

where $c = 5\pi/6\Gamma(1/3) \simeq 0.977$.

The z^2 dependence of the next-to-leading term in the limit (3-5) occurs not only for the choice (3-4) but for any $G(\Omega)$ that decreases more rapidly than Ω^{-3} as $\Omega\to\infty$. The dependence of the next-to-leading term in (3-5) is indicated in the Appendix to be plausible also on independent grounds. The dependence of (3-4) as $\Omega\to 0$ and the related dependence of (3-6) as $z\to\infty$ are open to question (see Appendix, Eq. (A-21), for a generalization of (3-6). These dependences, however, are irrelevant to all results to follow, except that numerical coefficients in Eqs. (3-7) and those following for α_m, a_m, and γ_m would be somewhat altered if the dependence in this limit were otherwise.

3.2 VELOCITY SPECTRUM IN THE MEAN REST FRAME

From the assumed intrinsic energy spectrum $\bar{E}_4(k,\omega)$ we can compute by (2-23) the spectrum $E_4(k,\omega)$ for the mean rest frame. We give approximate results based on two opposite limits. For the moment, we assume an approximately isotropic distribution $P(\vec{v})$. Eqs. (2-34) and (3-4) then yield

$$(3\text{-}8) \qquad E_4(k,\omega) = \frac{1}{2} a_m \epsilon^{1/2} k^{2(m-1)} \int_0^\infty dv v^2 P(v) \int_{-1}^{1} d\mu \frac{|\omega-kv\mu|^{-1/2}}{(k^2+\gamma_m^{\;2}\epsilon^{-1}|\omega-kv\mu|^3)^m}.$$

Two dimensionless variables are pertinent here, namely $\eta = \omega/v_o k$ and $\zeta \equiv \epsilon v_o^{-3} k^{-1} \sim (kL)^{-1}$, where the latter estimate follows from $\epsilon \sim v_o^3/L$. We may consider $\zeta << 1$, since we are considering the inertial subrange. In a domain where η is not too large, the convective approximation represents a valid lowest-order result; assuming a normal distribution for $P(v)$, inserting $E(k)$ in (2-95), and denoting this convective approximation by E_{4c}, we write

$$(3\text{-}9) \qquad E_{4c}(k,\omega) = (3/2\pi)^{1/2} N v_o^{-1} \epsilon^{2/3} k^{-8/3} \exp[-(3/2)(\omega/v_o k)^2]$$

with $N = [10/9\Gamma(1/3)]A_o$, in agreement with (2-95). By analysis of (3-8) we find the dispersive correction E_4-E_{4c} to lowest order in ζ. The result is given by

$$(3\text{-}10) \qquad E_4(k,\omega) \simeq E_{4c}(k,\omega)\{1-(3/2)b_o\alpha_m(\epsilon v_o^{-3}k^{-1})^{2/3}[1-3(\omega/v_o k)^2]\},$$

where $b_o = (3/5)\Gamma(1/3)/\Gamma(2/3) = 1.19$ [cf. (3-7)]. As we should expect,

dispersion thus decreases E_4 relative to E_{4c} for small $\omega/v_o k$ and increases it for larger $\omega/v_o k$. In a domain where $\omega/v_o k$ is large, as discussed previously, E_4 no longer is related to E_{4c} but is determined mainly by dispersion. From (3-8) in this instance, to lowest order in $(\omega/v_o k)^{-1}$ we obtain

$$E_4(k,\omega) \simeq a_m \epsilon^{3/2} \omega^{-7/2} (\epsilon k^2/\omega^3)^{m-1}, \tag{3-11}$$

the terms omitted being of relative order $(\omega/v_o k)^{-2}$. (We consider only $\omega \geq 0$, since $E_4(k,-\omega) = E_4(k,\omega)$.) The approximation (3-10) can be good only if $\zeta^{2/3} \ll 1$ and in addition, if $\eta \gtrsim 1$, the value (3-10) greatly exceeds (3-11). Hence, the domain of validity of (3-10) may be given roughly as that where

$$\zeta^{2/3} \ll 1 \text{ if } \eta \lesssim 1, \text{ or} \tag{3-12}$$

$$(\zeta\eta^{-3})^{m-1} \eta^{-7/2} \zeta^{5/6} \exp(3\eta^2/2) \ll 1 \text{ if } \eta \gtrsim 1; \tag{3-13}$$

similarly the domain of validity of (3-11) is that where

$$\eta \gg 1 \text{ and } (\zeta\eta^{-3})^{m-1} \eta^{-7/2} \zeta^{5/6} \exp(3\eta^2/2) \gg 1. \tag{3-14}$$

The result (3-11) corresponds to simply neglecting $kv\mu$ relative to ω and then $k^2/\gamma^2 \epsilon^{-1} \omega^3 = \gamma^{-2} \eta^{-3} \zeta$ relative to unity in the integrand of (3-8). Returning to the basic relation (2-23), we can take these steps more generally as follows:

$$E_4(\bar{k},\omega) = \int d^3\bar{v} P(\bar{v}) \tilde{E}_4(k,\omega - \bar{v}\cdot\bar{k}) \simeq \tilde{E}_4(k,\omega) \tag{3-15}$$

in the subject domain. The result (3-15) or (3-11) is evidently independent of the assumption of an isotropic $P(\bar{v})$. More generally, assuming only that $E_4(k,\omega)$ varies as $k^{2(m-1)}$ as $k \to 0$ at fixed ω/v_o, and recognizing that E_4 in this limit is determined by dispersion, not convection, and hence becomes independent of v_o, we can infer the result (3-11) (to within a constant), along with (3-15),

directly on dimensional grounds. Thus the result does not depend on the assumed explicit form (3-4) for $\tilde{E}_4(k,\omega)$ or on (2-6).

It is useful to extend the latter considerations also to the spectrum, say $\hat{E}_4(\bar{k},\omega)$ measured in a frame having a fixed velocity $-\bar{u}_o$ relative to the mean rest frame. $\hat{E}_4$ is given by the kinematic relation [Eq. (2-3)]

$$\hat{E}_4(\bar{k},\omega) = E_4(\bar{k},\omega - \bar{u}_o \cdot \bar{k}) . \tag{3-16}$$

We consider the domain of k,ω where conditions (3-14) are satisfied with ω replaced by $\omega - \bar{u}_o \cdot \bar{k} \equiv \omega'$, so that (3-11) and (3-15) hold for $E_4(k,\omega')$. Further restricting consideration to $k \ll \omega/u_o$, we then have from (3-16)

$$\hat{E}_4(\bar{k},\omega) \approx E_4(k,\omega) \simeq \tilde{E}_4(k,\omega) \simeq a_m \gamma_m^{-2m} \epsilon^{3/2} \omega^{-7/2} (\epsilon k^2/\omega^3)^{m-1} , \tag{3-17}$$

i.e., in the specified regime $\hat{E}_4$ is nearly independent of $\bar{u}_o$.

3.3 DISPERSIVE CORRECTION TO TAYLOR'S HYPOTHESIS

We proceed to consider the effect of dispersion on Taylor's ratio T, defined in Sec. 2.5. On assumption of our standard form (3-4) for $\tilde{E}_4(k,\omega)$, we may write the basic equation (2-103) as

$$\theta(\omega) = a_m \epsilon^{1/2} k^{2(m-1)} \int d^3\bar{v} P(\bar{v}) \int_o^\infty dk \int_{-1}^{1} d\nu \frac{|\omega - |\bar{u}_o + \bar{v}| k\nu|^{-1/2}}{(k^2 + \gamma^2 \epsilon^{-1} |\omega - |\bar{u}_o + \bar{v}| k\nu|^3)^m} \tag{3-18}$$

Likewise, the denominator of (2-100), by (2-47) and (2-101), is given by

(3-19) $$E^{(1)}(k_1) = (3/5)N\epsilon^{2/3}k_1^{-5/3}.$$

The ratio T is a function of two dimensionless variables, v_o/u_o and $\epsilon u_o^{-3}k_1^{-1}$. Initially, we regard these as independent and both small. The departure of T from unity, say ΔT, can then be approximated as the sum of a term ΔT_c computed by neglect of dispersion, with $\epsilon u_o^{-3}k_1^{-1} = 0$ (i.e., T - 1 as computed above in the local-convection approximation) and a term ΔT_d computed by neglect of local-convection, with $v_o/u_o = 0$. The actual domain of validity may be defined in the light of the results.

To find ΔT_d we take $v_o = 0$, whence (3-18) yields

(3-20) $$\hat{\theta}(\omega) \to a_m \epsilon^{1/2} k^{2(m-1)} \int_o^\infty dk \int_{-1}^1 d\nu \frac{|\omega - u_o k\nu|^{-1/2}}{(k^2 + \gamma^2 \epsilon^{-1}|\omega - u_o k\nu|^3)^m} .$$

Analysis of this expression to lowest order in $\epsilon u_o^{-3}k_1^{-1}$ yields, with some effort*,

(3-21) $$\Delta T_d = (5/3) b_m (B/A_o)^2 (\epsilon u_o^{-3} k_1^{-1})^{2/3},$$

where b_1 and b_3 were given at (3-10). In summary, we have

(3-22) $$T - 1 = \Delta T_c + \Delta T_d,$$

with ΔT_d given in the limit $v_o/u_o \to 0$ and to lowest order in $\epsilon u_o^{-3}k_1^{-1}$ by (3-21); from (2-108), we also have,

---

*If we consider instead T', defined by (2-100) with u' replacing u_o, then u_o^{-3} in (3-21) is replaced by $u_o^{-1}u'^{-2}$.

in the limit $\epsilon u_o^{-3}k_1^{-1} \to 0$,

$$\Delta T_c = \int d^3\bar{v}P(\bar{v})(|\bar{u}_o + \bar{v}|/u_o)^{2/3} - 1. \tag{3-23}$$

For isotropic normal $P(\bar{v})$, moreover, by (2-110) this becomes

$$\Delta T_c = [1 + s_o^2(v_o/u_o)^2]^{1/2}[1 - H_o(v_o/u_o)] - 1;$$

if also $v_o/u_o \ll 1$, by (2-111)

$$\Delta T \simeq (5/27)(v_o/u_o)^2.$$

Actually, $\epsilon u_o^{-3}k_1^{-1}$ is not independent of v_o/u_o, but rather $\epsilon u_o^{-3}k_1^{-1} \sim (v_o/u_o)^3(k_1L)^{-1}$, so that, by (3-21) ,

$$\Delta T_d \sim (v_o/u_o)^2(k_1L)^{-2/3}. \tag{3-24}$$

Since we are considering only the regime $k_1L \gg 1$, we have

$$\Delta T_d/\Delta T_c \sim (k_1L)^{-2/3} \ll 1. \tag{3-25}$$

Hence the local-convection approximation to this extent suffices, and the explicit results derived above on this basis are valid; furthermore, these hold independently of the magnitude of v_o/u_o. In particular, by the choice of convection velocities given in Sec. 2.5, Taylor's hypothesis becomes exact in the limit $1/k_1L \to 0$.

3 DISPERSIVE CORRECTION TO SPACE-TIME VELOCITY CORRELATIONS

We consider the dispersive corrections to the space-time decorrelations $\psi_1(\bar{r},\tau)$ computed in Sec. 2.3 by the local-convection approximation. For this purpose we assume the standard dispersive intrinsic spectrum $\tilde{E}_4(k,\omega)$ of (3-4). In the limits we shall consider, however, only the limiting result (3-5) is actually pertinent. Hence the results are independent of the specific form of (3-4), apart from the relation (3-7) of the constant α_m to B/A_o, and are contingent substantially only on the expected dependence $F(z)-F(0)\propto z^2$ as $z\to 0$.

The trace function $\psi(\bar{r},\tau)$ for the inertial subrange is given by Eq. (2-51). We assume an isotropic $P(\bar{v})$, whence

$$\psi(\bar{r},\tau) \equiv \psi(r,\tau) = \frac{1}{2}(\epsilon r)^{2/3} \int_0^\infty dv\, v^2 P(v) \int_{-1}^{1} d\mu\, \eta^{2/3} F(z\eta^{-2/3}), \tag{3-26}$$

where

$$\eta = [1+(v\tau/r)^2 - 2(v\tau/r)\mu]^{1/2}, \quad z = \epsilon^{1/3} r^{-2/3} |\tau| .$$

Eq. (3-26) is the equivalent of (2-69), but the dispersive factor $\sigma_o(z)$ in (2-56) is no longer replaced by unity. We assume again a normal $P(v)$. The ratio $\psi(r,\tau)/(\epsilon r)^{2/3}$ depends on the two variables $\beta_o = v_o\tau/r$ and z, or equivalently on β_o and

$$\lambda \equiv \beta_o^{-1} z^{3/2} = (\epsilon\tau)^{1/2}/v_o . \tag{3-27}$$

We have $\lambda \sim (v_o\tau/L)^{1/2} \ll 1$ by (2-5); λ provides an appropriate r-independent measure of the effect of dispersion.

The limit $\beta_o \to \infty$ is of particular interest, since, for $v_o/u_o \ll 1$, it determines the envelope of the space-time correlation curves $\hat{W}(\bar{r},\tau)$ in a measurement frame having velocity

$-\bar{u}_o$ (for $\bar{r}$ and $\bar{u}_o$ nearly parallel). By letting $\beta_o\to\infty$ and then $\lambda\to 0$, i.e., considering the limit where $\beta_o\to\infty$, $\lambda\to 0$, $\lambda\beta_o\equiv z^{3/2}\to\infty$, we are able to evaluate (3-26), by reference to (3-5) and (3-6), to the two lowest orders in β_o^{-1} with the two coefficients each evaluated to the two lowest orders in λ. Explicitly, we find

$$\text{(3-28)} \qquad \psi(r,\tau) \approx S_o(\epsilon v_o\tau)^{2/3}\left\{1+(1/3)\beta_o^{-2}+d_o\alpha_m\lambda^{4/3}[1-(1/3)\beta_o^{-2}]\right\},$$

where $d_o \approx (5/3)6^{-1/3}b_o^2 \approx 1.30$ and S_o was given at (2-76)*. Likewise, by considering the limit $\beta_o\to 0$ at arbitrary $\lambda(<1)$, we find from (3-26) and (3-5)

$$\text{(3-29)} \qquad \psi(r,\tau) \approx A_o(\epsilon r)^{2/3}\left\{1+(5/27)\beta_o^2+\alpha_m\lambda^{4/3}\beta_o^{4/3}[1-(1/27)\beta_o^2]\right\},$$

in which the two parts proportional to λ^o and $\lambda^{4/3}$ are evaluated to the lowest two orders in β_o.

Since the comparison with experiment considered here concerns the longitudinal decorrelation $\psi_1(r,\tau)$, we wish to obtain this also in approximations corresponding to (3-28) and (3-29) by reference to the continuity equation. The latter is still expressed by Eq. (2-78) in which, however, G_1 now depends on λ as well as on β_o. We find

$$\text{(3-30)} \qquad \psi_f(r,\tau) \approx S_1(\epsilon v_o\tau)^{2/3}\left\{1+(1/5)\beta_o^{-2}+d_o\alpha_m\lambda^{4/3}[1-(1/5)\beta_o^{-2}]\right\}$$

$$(\lambda\ll 1,\ \lambda\beta_o\gg 1),$$

where S_1 was given at (2-76); in the other limit

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*If, contrary to (3-5), $\tilde{\psi}(r,\tau)$ contained an additive component $B'\epsilon|\tau|$ for all τ, $\psi(r,\tau)$ would likewise contain this same r-independent component.

$$(3\text{-}31)\qquad \psi_f(r,\tau) \simeq A(\epsilon r)^{2/3}\Big\{1+(11/27)\beta_o^2+(11/7)\alpha_m\lambda^{4/3}\beta_o^{4/3} \times[1-(7/27)\beta_o^2]\Big\}\ (\beta_o \ll 1).$$

The transverse decorrelation $\psi_g(r,\tau)$ can be obtained similarly. Writing $\psi_f = \psi_{fc}+\Delta\psi_f$, where $\psi_{fc}(r,\tau)$ is the result of the local convection approximation and $\Delta\psi_f(r,\tau)$ the dispersive correction, however, we may insert the value of ψ_{fc} given by Eq. (2-80) or (2-84) without resort to the expansion of this part in (3- 30) and (3-31) in powers of β_o^{-1} and β_o, and use only the terms proportional to $\lambda^{4/3}$ in these equations as the expansion of $\Delta\psi_f$ alone.

We may consider the effect of the dispersive correction computed here on the comparison with experiment in Section 2.3 and Figure 2. For small angle ϕ between $\bar{r}$ and $\bar{u}_o$, the dispersive correction $\Delta\hat{\psi}_{11}(\bar{r},\tau)$ to $\hat{\psi}_{11}$ as given by (2-89) or (2-90) may be identified with $\Delta\psi_f(|\bar{r}-\bar{u}_o\tau|,\tau)$. We approximate the latter for $\hat{\beta}_o>1$ and $\hat{\beta}_o<1$, respectively, by the relevant terms in (3-30) and (3-31), where $\beta_o \equiv v_o\tau/|\bar{r}-\bar{v}_o\tau|$; this approximation is unjustified for $\bar{r},\tau$ such that $\beta_o\gtrsim1$ but $\lambda\beta_o\lesssim1$, but this domain is relatively small, and the envelope curve, in particular, will be given correctly. The additional parameter B/A_o now enters via α_m [recall Eqs. (3-7), (2-44), (2-45)].

The magnitude of the quantity λ of (3-27) in the experiment in question is given as a function of the

dimensionless time $n \equiv u_o\tau/M$, where M is the mesh spacing, according to (2-68) by $\lambda = (5/3)^{1/2}(v_o/u_o)Re_M^{1/2}Re_\lambda^{-1}n^{1/2}$, where $Re_M = u_oM/\nu$. For the parameters of the particular experiment, assuming $v_o/u_o = 0.027$, we find $\lambda = 0.116n^{1/2}$, which is moderately small, as required for such n that the correlation is high.

The space-time correlation computed by (3-30) and (3-31) for $\alpha_m = 2.31$ and $A = 0.55$ on assumption that $\emptyset=0$ and $v_o/u_o=0.027$ is compared in Figure 3 with the experimental results shown previously in Figure 2. The value used for α_m corresponds to $B/A_o=1.62$ if m=1 or $B/A_o=2.23$ if m=3 in Eq. (3-4). Also shown in Figure 3 (by solid lines) is the result obtained by supposing that the dispersive term $\Delta\psi_f$ is not that derived above but simply that corresponding to the space-independent (wavenumber-singular) form $\psi(r,\tau)-\psi(r,0) = B\epsilon|\tau|$, whence $\Delta\psi_f(r,\tau) = (1/3)B\epsilon|\tau|$; the result shown corresponds to $B/A_o = 1.04$ with other parameters as for the other computation. There is little to choose between the computed curves. The envelope in either case could be made to decrease more rapidly by increasing the assumed value of B/A_o. Figure 3 shows also the envelope curve (a), given previously in Figure 2, corresponding to $\emptyset = 0$, $v_o/u_o = 0.027$, but without dispersion ($B/A_o=0$). For likely values of B/A_o the dispersive effect is seen to influence the space-time correlations appreciably at the Reynolds numbers of the experiments considered.* Inclusion of dispersion necessarily worsens agreement with experiment for the envelope at the larger time delays, since the inertial-subrange form assumed already gives too small a correlation at such τ or $|\bar{r}-\bar{u}_o\tau|$ that the correlation is far from unity.

*Inclusion of dispersion does not affect the computed ratio ρ_c/ρ_s pertinent to shapes of isocorrelation curves discussed in Sec. 2.3.

4. PRESSURE CORRELATIONS AND SPECTRA

We turn to the subject of pressure fluctuations. Pressure is related to the velocity derivatives for an incompressible flow by the familiar Poisson equation

$$\nabla^2 p(\bar{x},t) = -\rho \partial^2 (v_i v_j)/\partial x_i \partial x_j, \tag{4-1}$$

where $v_i(\bar{x},t)$ refers to the total velocity or, in the unsheared flows being considered, the fluctuating velocity. Prior to the common pursuit of consequent relations, we progress by simpler considerations.

4.1 PRESSURE SPECTRA IN A MOVING FRAME BY KINEMATICS AND SIMILARITY FOR THE INERTIAL SUBRANGE

We consider once more a frame having constant velocity $-\bar{u}_o$ relative to the mean rest frame and refer to the pressure on a planar area whose normal is orthogonal to $\bar{u}_o$. Let the spectral density of pressure in frequency ω and two-component wave number $\bar{K} = (k_1,k_3)$ in the plane containing this area be denoted by $\hat{P}(\bar{K},\omega)$. In the instance of the rest frame ($\bar{u}_o$=0), we denote this spectrum by $P(\bar{K},\omega)$. We then have*

$$\hat{P}(\bar{K},\omega) = P(\bar{K},\omega - \bar{u}_o \cdot \bar{K}) . \tag{4-2}$$

We assume that the large eddies are roughly isotropic,

*In the terminology of Chandiramani (1965) and others, $\hat{P}(\bar{K},\omega)$ (more precisely its wavenumber integral) is the fixed-transducer spectrum and $P(\bar{K},\omega)$ the moving-axis spectrum.

so that the local fluctuating velocity is adequately characterized by v_o, and limit consideration to the inertial subrange $L^{-1} << (K^2+\omega^2/u_o^2)^{1/2} << \ell_o^{-1}$ $(u_o \gg v_o)$, where $K = |\bar{K}|$. $\hat{P}(\bar{K},\omega)$, as a function of k_1, k_3, at fixed ω, has a peak, or rather a ridge, in region of $\bar{K}$ where the probability of a total fluid velocity, $\bar{u}_o + \bar{v}$, that satisfies the convective condition $\omega-(\bar{u}_o+\bar{v})\cdot\bar{K} = 0$ is substantial; this region is given by

$$(4\text{-}3) \qquad |\omega-\bar{u}_o\cdot\bar{K}| \lesssim v_o K.$$

We recall in the parallel instance of the mean-rest-frame velocity spectrum $E_4(k,\omega)$ that the result of the local-convection approximation, $E_{4c}(k,\omega)$ [see(3-9)], was adequate [see((3-10)] if $\omega/v_o k \lesssim 1$ (or even somewhat larger, according to (3-13), if ζ is sufficiently small). Similarly here, we expect that a local-convective approximation to the mean-rest-frame pressure spectrum $P(\bar{K},\omega')$ will be adequate roughly if $\omega'/v_o K \lesssim 1$. Now, the pressure is quadratically related to the velocity fluctuations, and the velocity spectrum (or correlation) in the inertial subrange is proportional to $\epsilon^{2/3}$. Hence $P(\bar{K},\omega')$ in the local-convection domain, being otherwise independent of velocity dispersion and thus of ϵ, must have the form

$$P(\bar{K},\omega') = \rho^2\epsilon^{4/3}D(v_o,\bar{K},\omega'),$$

where D is a function having dimensions (length)$^{10/3}$ (time). Since D must also be a properly convariant scalar function of the vector $\bar{K}$, we must have

$$D(\hat{v},K,\omega') = v_o^{-1} K^{-13/3} F(v_o\bar{K}/\omega');$$

on account of isotropy, however, F can depend on $\bar{K}$ only via K, say $F \equiv \phi(\omega'/v_oK)$. Thus, in the subject domain, we have

(4-4) $$P(\bar{K},\omega') = \rho^2\epsilon^{4/3} v_o^{-1} K^{-13/3} \phi(\omega'/v_oK), \text{ whence}$$

(4-5) $$\hat{P}(\bar{K},\omega) = \rho^2\epsilon^{4/3} v_o^{-1} K^{-13/3} \phi([\omega-\bar{u}_o\cdot K]/v_oK)$$

We now assume $u_o \gg v_o$, choose coordinates such that $\bar{u}_o = (u_o,0,0)$, and consider the width Δk_1 in k_1 of the convective ridge in $\hat{P}(\bar{K},\omega)$ centered at $k_1 \simeq \omega/u_o$. According to (4-3), we have $\Delta k_1 \sim (v_o/u_o)K$. The factor $K^{-13/3}$ in (4-5) with $K^2 \simeq (\omega/u_o)^2 + k_3^2$ effectively cuts off contributions from $k_3 \gtrsim \omega/u_o$. Hence $\Delta k_1 \lesssim \sqrt{2}(v_o/u_o)(\omega/u_o)$ for all significant k_3, so that $\Delta k_1/k_1 \sim v_o/u_o \ll 1$. Hence, when multiplied by a power or other sufficiently smooth function of K, the function ϕ be behaves roughly as a δ-function:

(4-6) $$\phi([\omega-u_ok_1]/v_ok) \rightarrow a_\phi(v_o/u_o)K\delta(k_1-\omega/u_o),$$

where $a_\phi = \int_{-\infty}^{\infty} dx\phi(x)$. This approximation is equivalent to Taylor's hypothesis (based on the unmodified convection velocity u_o).

We may now consider the frequency spectrum of the point pressure,

(4-7) $$\hat{P}(\omega) = \int d^2\bar{K}\hat{P}(\bar{K},\omega),$$

where the integration runs over the entire $\bar{K}$-plane. Since in this case $\hat{P}(\bar{K},\omega)$ is weighted uniformly in $\bar{K}$, $\hat{P}(\omega)$ is largely

determined by the domain of $\bar{K}$ which contains the peak in $\hat{P}(\bar{K},\omega)$ and where (4-5) holds. From (4-5) to (4-7) we readily obtain

$$\hat{P}(\omega) = c_{5/3} a_{\phi} \rho^2 \epsilon^{4/3} u_o^{-1} (\omega/u_o)^{-7/3}, \tag{4-8}$$

where

$$c_n = 2 \int_0^\infty dx(x^2+1)^{-n} = \pi^{1/2}[\Gamma(n-1/2)/\Gamma(n)]. \tag{4-9}$$

We thus infer uniquely the frequency dependence, $\hat{P}(\omega) \propto \omega^{-7/3}$. The conditions for (4-8) are $v_o/u_o \ll 1$, $\omega L/u_o \gg 1$, and $\omega \ell_o/v_o \ll 1$.*

We consider now the spectrum, $\hat{Q}(\omega)$, of average pressure on a circular area of radius R_o, given by

$$\hat{Q}(\omega) = \int d^2\bar{K}[2J_1(KR_o)/KR_o]^2 \hat{P}(\bar{K},\omega). \tag{4-10}$$

This quantity, for the homogeneous turbulence assumed, is not actually accessible to experiment, since the introduction of a pressure sensor of the prescribed area would naturally disturb the homogeneity of the flow near the surface of the sensor. We form the quantity rather as an edifying analog for the spectrum of average pressure on a similar area of a wall in the more difficult case of a turbulent boundary layer.

Suppose first $\omega R_o/u_o \lesssim 1$, so that we may approximate the area-averaging factor by

$$[2J_1(KR_o)/KR_o]^2 \simeq 1 - (1/4)(KR_o)^2.$$

*By (2-40), the last condition may be written $(\omega\nu/v_o^2)(v_oL/\nu)^{1/4} \ll 1$.

Then (4-5), (4-6), (4-10) and (4-8) yield

(4-11) $$\hat{Q}(\omega) \approx \hat{P}(\omega)[1-(\omega R_o/u_o)^2].$$

Consider now the opposite limit $\omega R_o/u_o \gg \pi$. In this case the range of K near the peak of $\hat{P}(\bar{K},\omega)$ in (4-10), namely $K \sim \omega/u_o$, is well separated from the range $K \lesssim 2\pi R_o^{-1}$ where the area-averaging factor is relatively large. We must examine the contribution to $\hat{Q}(\omega)$ from the latter range, say $\hat{Q}_-(\omega)$, as well as that from the former, say $\hat{Q}_+(\omega)$.

In the range of K pertinent to $\hat{Q}_-$, since $K \ll \omega/u_o$, we may approximate $\hat{P}(\bar{K},\omega)$ by the rest-frame spectrum $P(\bar{K},\omega)$. Form (4-4) for $P(\bar{K},\omega)$ may not hold for the range of $\bar{K}$ where $K \ll \omega/v_o$, depending on the role of dispersion. At the same time, we have no justification for presuming that $P(\bar{K},\omega)$ here does not depend explicitly on v_o and is independent of K, as for m = 1 in the instance of the form (3-11) for the velocity spectrum $E_4(k,\omega)$ (if these conditions held we could uniquely infer $P(\bar{K},\omega) \sim \rho^2\epsilon^2\omega^{-6}$). In Sec. 4.2 it is indicated that in this domain, even though $\omega/v_oK \gg 1$, $P(\bar{K},\omega)$ in fact is likely determined by local convection and assumes the K-independent form of (4-14):

(4-12) $$P(\bar{K},\omega) \sim A_p\rho^2\epsilon^{4/3}v_o^{10/3}\omega^{-13/3},$$

where A_p is a constant of the order of unity. The range of K where (4-12) applies then yields, in the limit where $\omega R_o/v_o \gg 1$, the largest contribution to the integral of form (4-10) for $\hat{Q}_-(\omega)$ in the sum

(4-13) $$\hat{Q}(\omega) \simeq \hat{Q}_-(\omega)+\hat{Q}_+(\omega).$$

From (4-12) we thus obtain

(4-14) $$\hat{Q}_-(\omega) \simeq 4\pi A_p\rho^2\epsilon^{4/3}v_o^{10/3}\omega^{-13/3}R_o^{-2}$$

for $\omega R_o/u_o \gg 1$, $\omega R_o/v_o \gg 1$, $\omega \ell_o/v_o \ll 1$.

To find $\hat{Q}_+$ we may assume for $\hat{P}(\bar{K},\omega)$ in the pertinent region of $\bar{K}$ the earlier form (4-5) and, without appreciable error, extend the integral of the form (4-10) for $\hat{Q}_+$ over all $\bar{K}$. Treating separately two opposite limiting cases, $(\omega R_o/u_o)(v_o/u_o) \gg \pi$ and $\ll \pi$, using (4-6), and at the proper stages making use of the result $(4/\pi)(KR_o)^{-3}$ for the average of $[2J_1(KR_o)/KR_o]^2$ over an interval ΔK such that $\pi \ll \Delta K \ll K$, we find both cases

(4-15) $$Q_+(\omega) \simeq 4\pi^{-1} c_{19/6} a_\phi \rho^2 \epsilon^{4/3} u_o^{-1} (\omega/u_o)^{-16/3} R_o^{-3}$$

with c_n given at (4-9). Conditions for (4-15) and (4-11) (apart from those on $\omega R_o/u_o$) are as given for Eq. (4-8).

$\hat{Q}_-$ varies as R_o^{-2}, corresponding to a nonvanishing effective area scale, whereas $\hat{Q}_+$ varies as R_o^{-3}. Their ratio, by (4-14) and (4-15), is

(4-16) $$\hat{Q}_-/\hat{Q}_+ \simeq a_o (v_o/u_o)^{10/3} (\omega R_o/u_o),$$

where $a_o = \pi^2 A_p / c_{19/6} a_\phi$.* As for the reduction factor due to area averaging, by (4-8) we have

(4-17) $$\hat{Q}(\omega)/\hat{P}(\omega) = \hat{q}_-(\omega) + \hat{q}_+(\omega),$$

$$\hat{q}_- \equiv \hat{Q}_-/\hat{P} \simeq a_- (v_o/u_o)^{10/3} (\omega R_o/u_o)^{-2},$$

$$\hat{q}_+ \equiv \hat{Q}_+/\hat{P} \simeq a_+ (\omega R_o/u_o)^{-3},$$

where $a_- = 4\pi A_p / c_{5/3} a_\phi$ and $a_+ = 4c_{19/6}/\pi c_{5/3} \simeq 0.795$.

For arbitrary values of $\omega R_o/U_o$, we may infer for $\hat{Q}_+(\omega)$, from (4-5) and (4-10), a functional form

(4-18) $$\hat{Q}_+(\omega) = \rho^2 \epsilon^{4/3} u_o^{-1} R_o^{7/3} h(v_o/u_o, \omega R_o/u_o).$$

*Analogously, for boundary-layer turbulence we may conjecture that $Q_-/Q_+ \sim (v_*/U_\infty)^2 (\omega R_o/U_\infty)$.

By (4-8) and (4-15), limiting forms of the function $h(x,y)$ are given by $h(x,y) \to \text{const.} \, X \cdot y^{-7/3}$ as $x \to 0$, $y \to 0$ and $h(x,y) \to \text{const.} \, X \, y^{-16/3}$ as $x \to 0$, $y \to \infty$. Taking account of the relation $\epsilon \approx v_o^3/L$, we may make explicit all velocity dependence in the scaling form (4-18) by writing

$$\hat{Q}_+(\omega) \sim \rho^2 R_o u_o^3 (\omega L/u_o)^{-4/3} w(v_o/u_o, \omega R_o/u_o),$$

where $w(x,y)$ is a function related to $h(x,y)$.

4.2 PRESSURE SPECTRA AND CORRELATIONS BY THE MODEL OF QUASINORMAL VELOCITY DISTRIBUTIONS

The results of the preceding section can be clarified and extended in part, at the expense of introducing imprecisely assessed approximations, by pursuing an explicit calculable model to give $\hat{P}(\bar{K},\omega)$. The model to be considered will have approximate validity only in the convective range (4-3) or somewhat beyond, and not in the dispersive range, in particular not where (4-13) applies.

From the basic relation (4-1) of the pressure field to its velocity-derivative sources, the rest-frame spectrum $P(\bar{K},\omega)$ is found to be expressible in the present case of stationary infinite homogeneous turbulence as the following double integral over the coordinate normal to the plane of $\bar{K}$ (e.g., from Kraichnan 1956):

(4-19) $$P(\bar{K},\omega) = (1/4)\rho^2 K^{-2} \int_{-\infty}^{\infty} dx_2 \int_{-\infty}^{\infty} dx_2' \exp[-K(|x_2|+|x_2'|)] \times S(x_2' - x_2,\bar{K},\omega),$$

where

(4-20) $$S(\zeta_2,\bar{K},\omega) = (2\pi)^{-3} \int d^2\bar{\zeta} \int d\tau \exp[-i(\bar{K}\cdot\bar{\zeta} - \omega\tau)] Q(\zeta_2,\bar{\zeta},\tau),$$

(4-21) $\quad Q(\zeta_2,\bar{\zeta},\tau) = [\partial_\alpha\partial_\beta\partial'_\mu\partial'_\nu(\langle v_\alpha v_\beta v'_\mu v'_\nu\rangle - \langle v_\alpha v_\beta\rangle\langle v'_\mu v'_\nu\rangle)]_{\bar{x}'=\bar{x}+\bar{r}}$,

$v_\alpha = v_\alpha(\bar{x},t)$, $v'_\alpha = v_\alpha(\bar{x}',t+\tau)$, $\bar{x} = (x_1,x_2,x_3)$, $\bar{r} = (\zeta_1,\zeta_2,\zeta_3)$ $\bar{\zeta} = (\zeta_1,\zeta_3)$, $\partial_\alpha \equiv \partial/\partial x_\alpha$, $\partial_\alpha' \equiv \partial/\partial x'_\alpha$, and integrals extend over the infinite domain unless otherwise specified. $P(\bar{K},\omega)$ as given by (4-19) - (4-21) is so normalized that the mean squared fluctuating pressure is given by

(4-22) $\quad \langle p^2\rangle = \int_{-\infty}^{\infty}d\omega P(\omega) = \int_{-\infty}^{\infty}d\omega\int d^2\bar{K}P(\bar{K},\omega)$.

We now assume the turbulence is isotropic; regarding the inertial subrange, this assumption corresponds again to isotropic $P(\bar{v})$ in (2-6). In this event we may write Q of (4-21) as $Q(r,\tau)$. We define $L(k,\omega)$ as the wavenumber-frequency spectrum of Q:

(4-23) $\quad L(k,\omega) = \int d^3\bar{r}\int d\tau \exp[-i(\bar{k}\cdot\bar{r} - \omega\tau)]Q(r,\tau)$

$[\bar{k} = (k_1,k_2,k_3)]$. Then S in (4-19) may be written as a transform of L over k_2. In the present instance (unlike that of boundary-layer turbulence) we can trivially integrate over x_2 and x'_2 in (4-19) to obtain

(4-24) $\quad P(\bar{K},\omega) = (2\pi)^{-4}\rho^2\int dk_2 k^{-4}L(k,\omega)$,

in which, we recall, $k^2 = k_2^2 + K^2$.

To proceed further we make the common assumption of a quasinormal velocity distribution with reference to the two-point, four-component space-time correlation $\langle v_\alpha v_\beta v'_\mu v'_\nu\rangle$ in Eq. (4-21) for $Q(r,\tau)$, i.e. we assume that the fourth-order moments are related to the second-order moments in the same way as if the statistical distribution of velocity in space-time were a joint normal distribution (Millionshtchikov 1944,

Batchelor 1951). This assumption has received serious criticism (Kraichnan 1957, 1959a) as the basis for a dynamic treatment of turbulence. Nevertheless, its use appears unlikely to lead to major error in the less demanding application to a merely descriptive treatment of velocity and pressure correlations, and the assumption has often been so used. It was first employed in the context of pressure fluctuations by Obukhov (1949). From this assumption, Batchelor (1951) derived the form of the desired correlation function $Q(r,\tau)$ for vanishing time separation, i.e. $Q(r,0)$. If the assumption of quasinormality is made also for $\tau \neq 0$ (Chandrasekhar 1955), the derivation applies equally when the two points in question are separated in time as well as space. In terms of the (mean-rest-frame) longitudinal velocity decorrelation $\psi_1(r,\tau)$ the relation derived is

(4-25) $$Q(r,\tau) = 4[2(\psi_1'')^2 + 2\psi_1'\psi_1''' + 10r^{-1}\psi_1'\psi_1'' + 3r^{-2}(\psi')^2],$$

where a prime denotes $\partial/\partial r$. Alternatively, the spatial transform of $Q(r,\tau)$ defined by

$$q(k,\tau) = (2\pi)^{-3}\int d^3\bar{r}e^{-i\bar{k}\cdot\bar{r}}Q(r,\tau) = \frac{1}{2}\pi^{-2}k^{-1}\int_0^\infty dr\,r\sin kr\,Q(r,\tau)$$

is related to the corresponding transform $E(k,\tau)$ of $\psi(r,\tau)$ (see Eq. (2-17)) by

(4-26) $$q(k,\tau) = (8\pi^2)^{-1}k^4\int d^3\bar{k}'E(k',\tau)E(|\bar{k}-\bar{k}'|,\tau)\sin^4\theta'|\bar{k}-\bar{k}'|^{-4},$$

where $\cos\theta' = \bar{k}\cdot\bar{k}'/kk'$. $L(k,\omega)$ of (4-23) is then given as a time transform of $q(k,\tau)$. By virtue of its definition (4-21) as a fourth-order derivative, $Q(r,\tau)$, as shown for $\tau = 0$ by Batchelor (1951), corresponds to a $q(k,\tau)$ and hence to a spectrum $L(k,\omega)$ that vary, for any reasonable function $\psi_1(r,\tau)$,

as k^4 for k<<1, and likewise in the case where these quantities are generalized to anisotropy at low wave numbers. From (4-24), (4-26) and (2-22) we can express $P(K,\omega)$ also in terms of $E_4(k,\omega)$:

(4-26a) $$P(\bar{K},\omega) = (1/8\pi^2)\rho^2\int_{-\infty}^{\infty} dk_2 \int d^3\bar{k}' \sin^4\theta' |\bar{K}-\bar{k}'|^{-4}$$
$$\times \int_{-\infty}^{\infty} d\omega' E_4(k',\omega')E_4(|\bar{K}-\bar{k}'|,\omega-\omega')$$

In principle, in the inertial subrange we can assume our standard form (3-4) for the intrinsic energy spectrum $\tilde{E}_4(k,\omega)$ (suitably extended to low k or terminated at $k\sim L^{-1}$), write the resulting $E_4(k,\omega)$ as in (3-8), or more generally (2-34), and compute the desired $P(\bar{K},\omega)$ from (2-26a) or else proceed via (4-25) or (4-26). We shall deal separately with the domain $\omega/v_o K>>1$ and the opposite domain where dispersion may be neglected, starting with the latter.

In the non-dispersive domain we can simply approximate $E(k,\tau)$ by (2-96) or $\Psi_1(r,\tau)$ by (2-80). On account of the still considerable analytic difficulty, however, and the uncertain error already introduced by the assumption of quasinormality, we proceed still more crudely. We approximate $\Psi_1(r,\tau)$ in (4-25) as a space-time isotropic function. The consequent error in the computed $P(\bar{K},\omega)$ for the inertial subrance relative to an exact local-convective approximation will parallel the difference previcusly noted between (2-96) and (2-97) in the case of the energy spectrum $E_4(k,\omega)$; furthermore, dispersion will limit the validity even of an exact local-convection approximation for $P(\bar{K},\omega)$ in a parallel manner to that discussed for $E_4(k,\omega)$ in Sec. 3.2.

In the stated approximation, we have

(4-27) $$\Psi_1(r,\tau) \simeq \Psi_1(R_1,0) \equiv \Psi_1(R_1)$$

with R_1 of the form (2-81). In the inertial subrange, where, as at (2-84),

(4-28) $$\psi_1(R_1) = A(\epsilon R_1)^{2/3} \qquad (\ell_o \ll R_1 \ll L)$$

with R_1 given specifically by (2-81) and (2-83), we find from (4-25)

(4-29) $$Q(r,\tau) = (16/9)A^2\epsilon^{4/3}R_1^{-8/3}[15-(80/3)r^2/R_1^2+(112/9)r^4/R_1^4].$$

In the viscous subrange, where, as at (2-86),

$$\psi_1(R_1) = (\epsilon/30\nu)R_{o1}^2 \qquad (R_{o1} \lesssim \ell_o)$$

with R_{o1} given by (2-87) and (2-88), we find

(4-30) $$Q(r,\tau) = 4(\epsilon/15\nu)^2(14 + r^4/R_{o1}^4).$$

We note further the result where a modified von Karman interpolation form (Hinze 1959, Eq.(3-131))

(4-31) $$\psi_1(R_1) = (1/3)v_o^2[1-\exp(-k_oR_1)]$$

is assumed, in which k_o is a constant inverse correlation length:

(4-32) $$Q(r,\tau) = (4/9)v_o^4\exp(-2k_oR_1)k_o^2[4k_o^2(r^4/R_1^4)-10k_oR_1^{-1}$$
$$\times(r^2/R_1^2)(2-r^2/R_1^2)+R_1^{-2}(15-20r^2/R_1^2+8r^4/R_1^4)].$$

For the inertial-subrange form (4-29), the integral (4-23) can be performed to yield, in a corresponding range, the function $L(k,\omega)$ in (4-24). We find

(4-33) $$L(k,\omega) = A_Q^2\, v_e^{-1}\epsilon^{4/3}k^4k_+^{-16/3} \qquad (L^{-1} \ll k_+ \ll \ell_o^{-1}),$$

where

$$(4\text{-}34)\qquad k_+^2 = k^2 + (\omega/v_e)^2, \qquad v_e = s_1 v_o,$$

$$A_Q^2 = 2^{4/3}(640/81)\,\pi^2[\Gamma(2/3)/\Gamma(4/3)]A^2 .$$

Before proceeding, we formally generalize to the non-universal range where the condition $k_+ \gg L^{-1}$ is not satisfied. We continue to use the local-convection and space-time isotropic approximations as at (4-27). There is no justification for this extension, but it may prove qualitatively useful to point out the resulting difference. If the reciprocal size of the energy-containing eddies is $\sim k_o(\sim L^{-1})$, it is natural to suppose that the alteration in $L(k,\omega)$ that would result from using a $\psi_1(R_1)$ roughly appropriate also where $k_+ \lesssim k_o$ would be to replace k_+ in (2-54) by something like $(k_+^2+4k_o^2)^{1/2}$, where the factor on k_o^2 is uncertain but set equal to four for reasons to appear.

To examine the point explicitly, we may employ (4-31) and (4-32). Form (4-31) corresponds to an energy spectrum $E(k)$ that behaves appropriately, i.e., as k^4, for $k \ll k_o$, whereas (4-28) corresponds to one that varies as $k^{-5/3}$ for all k. Form (4-31) assumes that the turbulence is isotropic, however, even in the energy-containing range. It does not reduce to form (4-28) for $R_1 \ll L$(or $k_+ \gg k_o$), as would the unmodified von Karman interpolation form (Hinze 1959, (Eq. 3-136)), but suffices for the present purpose. Inserting (4-32) in (4-23), we find after considerable labor*

*It is convenient to evaluate (4-23) by use of four-dimensional spherical coordinates.

$$(4\text{-}35)\qquad L(k,\omega) = (2\pi/3)^2 v_o^4 v_e^{-1} a^2 \times \frac{k^4[40a^4+45a^2k_+^2+8k_+^4+5a(k_+^2+a^2)^{1/2}(8a^2+5k_+^2)]}{(k_+^2+a^2)^{5/2}[(k_+^2+a^2)^{1/2}+a]^5}$$

with $a \equiv 2k_o$. In the high-k limit where $\omega/v_c k \rightarrow 0$ and $a/k \rightarrow 0$, (4-35) becomes

$$(4\text{-}36)\qquad L(k,\omega) \rightarrow (32\pi/9)v_e^{-1}v_o^4 a^2 k^{-2} \;;$$

similarly, in the limit where $a/k_+ \rightarrow 0$, it becomes

$$(4\text{-}37)\qquad L(k,\omega) \rightarrow (32\pi/9)v_e^{-1}v_o^4 a^2 k^4 k_+^{-6},$$

which may be compared with the unmodified inertial-subrange result (4-33) that pertains to the same limit. A simplified form that approaches the same limits (4-36) and (4-37) and broadly preserves the character of (4-35) in the general domain of k and ω is given by

$$(4\text{-}38)\qquad L(k,\omega) = (32\pi/9)v_e^{-1}v_o^4 a^2 k^4 (k_+^2+a^2)^{-3} \;.$$

In view of the approximations already made and the non-universality in the range in question, we may accept (4-38) in place of (4-35). Since, however, (4-37) and (4-38) justify the initially suggested replacement $k_+^2 \rightarrow k_+^2+a^2$ for general a/k_+, we can preserve the universal character for $k_+ \gg a$ by accepting not (4-38) but the generalized form of (4-33):

$$(4\text{-}39)\qquad L(k,\omega) \simeq A_Q^2 v_e^{-1} \epsilon^{4/3} k^4 (k_+^2+a^2)^{-8/3},$$

where $2a^{-1}$ roughly measures the size of the energy-containing eddies.

Inserting (4-39) in (4-24), we obtain the wavenumber-frequency spectrum of pressure in the mean rest frame:

(4-40) $$P(\bar{K},\omega) = C_p^2\rho^2\epsilon^{4/3}v_e^{-1}K_t^{-13/3},$$

where

$$K_t^2 = K^2+(\omega/v_e)^2+a^2,$$

$$C_p^2 = (8/9)2^{1/3}\pi^{-3/2}[\Gamma(13/6)/\Gamma(4/3)]A^2.$$

To obtain from (4-40) the corresponding spectrum $\hat{P}(K,\omega)$ in the measurement frame having relative velocity $-\bar{u}_o$, we need only use (4-2) to find

(4-41) $$\hat{P}(K,\omega) = C_p^2\rho^2\epsilon^{4/3}v_e^{-1}\hat{K}_t^{-13/3},$$ where

(4-42) $$\hat{K}_t^2 \equiv K^2+(\omega-u_ok_1)^2/v_e^2+a^2$$

and we choose $\bar{u}_o = (u_o,0,0)$. On account of our use of the local-convection approximation, the result for $\hat{P}(\bar{K},\omega)$ can hold only where $|\omega-u_oK_1|/v_e \lesssim K$, corresponding to the region of the convective peak (4-3). Comparing (4-40) for $\hat{P}(\bar{K},\omega)$ with (4-4) (and setting $a=0$ for the inertial subrange) we see that according to the present model, the function ϕ there introduced is given by

(4-43) $$\phi(x) = C_p^2 s_1^{-1}(1 + s_1^{-2}x^2)^{-13/6}.$$

In view of the previous comparison between (2-96) and (2-97) for $E_4(k,\omega)$, however, we should expect that an exact calculation of the local convection approximation for $P(\bar{K},\omega)$ by means of (4-26) (still on the basis of a normal isotropic velocity distribution) would yield a function $\phi(x)$ different from (4-43); in any case, the result is not to be used for $x\gg1$. Eq. (4-43) yields

for the value of the coefficient a_ϕ defined at (4-6)

$$a_\phi = C_p^2 c_{13/6}, \tag{4-44}$$

where $c_{13/6}$ is given by (4-9).

In the present model, the frequency spectrum $\hat{P}(\omega)$ can be found exactly from (4-7) for arbitrary v_o/u_o; from (4-40) with $K_t \to \hat{K}_t$, we obtain

$$\hat{P}(\omega) = (6\pi/7) C_p^2 \rho^2 \epsilon^{4/3} u_e^{-1} (\omega^2/u_e^2 + a^2)^{-7/6}, \tag{4-45}$$

where u_e is the sum of the mean convection speed u_o and the effective fluctuating convection speed v_e in quadrature, given at Eq. (2-120) ($v_e = s_1 v_o$). In the limit $v_o/u_o \to 0$ (with $a=0$), this reduces to the form (4-8) obtained in this limit more generally. Eq. (4-45) requires the nonviscous condition $\omega \ell_o / v_o \ll 1$.

We may consider Taylor's hypothesis for pressure, as in Sec. 3.3 for velocity, by regarding the ratio

$$T_p \equiv u_p \hat{P}(u_p k_1)/P_1(k_1), \tag{4-46}$$

where $P_1(k_1)$ is the mean-rest-frame spectrum in streamwise wave number at a fixed time and cross-stream position (x_3):

$$P_1(k_1) = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_3 P(\bar{K},\omega),$$

and u_p is an effective convection velocity open to choice. From (4-41) and (4-45) we find $T_p = 1$ exactly provided $u_p = u_e$, given at (2-120).

By (4-22), Eq. (4-45) yields a mean squared pressure

$$\langle p^2 \rangle = (8/9) 2^{1/3} [\Gamma(2/3)/\Gamma(4/3)] A^2 \rho^2 \epsilon^{4/3} a^{-4/3},$$

properly independent of u_o and in accord with the usual order-of-magnitude relation $\langle p^2 \rangle^{1/2} \sim \rho \langle v_o^2 \rangle$.

Concerning the properties of the convective ridge in $\hat{P}(\bar{K},\omega)$ as given by (4-40), at fixed k_3 the quantity $\hat{K}_t^2$ has

its minimum and hence $\hat{P}(\bar{K},\omega)$ its maximum at

(4-47) $$k_1 = k_{1m} \equiv (\omega/u_o)(u_o/u_e)^2 = \omega/u_c \;;$$

this minimum value of $\hat{K}_t^2$ is given by

$$\hat{K}_{tm}^2 = (\omega/u_e)^2 + k_3^2 + a^2;$$

the half width Δk_1 of $\hat{P}(\bar{K},\omega)$ at one-fourth maximum is given for $\Delta k_1 < k_{1m}$ by

(4-48) $$\Delta k_1 = (v_e/u_e)(\omega^2/u_e^2 + k_3^2 + a^2)^{1/2}$$

(cf. paragraph following (4-4)). The contours of constant $\hat{P}(\bar{K},\omega)$ in $\bar{K}$-space are defined by $\hat{K}_t^2$ = constant; by (4-42) they are ellipses centered at $k_1 = \pm k_{1m}$, $k_3 = 0$ and have semiaxes along k_1 and k_3 in the ratio v_e/u_e .

From (4-41), (4-47), and (4-48) we see how, as u_o increases from zero, the ridge in $\hat{P}(\bar{K},\omega)$ sharpens and moves from $k_1 = 0$ out to $k_1 = \omega/2v_e$ at $u_o = v_e$ and then again toward lower k_1; similarly, from (4-45) we see how, as u_o increases, the peak in $\hat{P}(\omega)$ at $\omega = 0$ becomes broader and lower, with $\hat{P}(\omega)$ decreasing for $\omega < (\sqrt{3}/2)au_e$ and increasing for $\omega > (\sqrt{3}/2)au_e$.

We consider now the frequency transform of the spatial correlation (cross-spectral density) in the measurement frame,

(4-49) $$\hat{\theta}_p(\bar{\tau},\omega) = \int d^2\bar{K}\hat{P}(\bar{K},\omega)e^{i\bar{K}\cdot\bar{\tau}} .$$

Defining the normalized magnitude M and effective convective velocity u_c by setting

(4-50) $$\hat{\theta}_p(\bar{\zeta},\omega) = M(\bar{\zeta},\omega)\hat{P}(\omega)\exp(i\omega\zeta_1/u_c),$$

from (4-40) and (4-45) we find that u_c is again as given by Eq. (2-120) and

$$(4\text{-}51) \qquad M(\vec{\zeta},\omega) = [2^{1/6}\Gamma(7/6)]^{-1}(\omega^2/u_e^2+a^2)^{7/12}(\zeta_3^2+\nu_e^2\zeta_1^2)^{7/12}$$

$$\times\ K_{7/6}((\omega^2/u_e^2+a^2)^{1/2}(\zeta_3^2+\nu_e^2\zeta_1^2)^{1/2}),$$

where $\nu_e = v_e/u_e$ as at (2-120). The elliptical contours of constant $M(\vec{\zeta},\omega)$ have axes along ζ_1 and ζ_3 in the ratio u_e/v_e inverse to that pertaining to $\hat{P}(\vec{K},\omega)$; this ratio gives also the relative scales of the principal coefficients $A(\zeta_1,\omega) \equiv M(\zeta_1,0,\omega)$, $B(\zeta_3,\omega) \equiv M(0,\zeta_3,\omega)$.

Taking the liberty of comparing with experimental results for boundary-layer, as opposed to homogeneous, flow, with the rms fluctuating velocity v_o in $v_e = s_1 v_o$ attributed a value characteristic of the constant-stress layer, say $v_o = 0.1u_o$ in the regime where the friction velocity v_* is given by $v_*/u_o \simeq 0.033$, we obtain for u_e/v_e a value in good agreement with the experimentally observed ratio of scales ($\simeq 7.5$) (Corcos 1963, Willmarth and Wooldridge 1962). Other rough correspondences are evident between present results (e.g., (4-45), (4-54)) and measurements for boundary-layer turbulence; in view of the great differences between these types of flow these correspondences will not be pursued, pending a careful extension of the present approach.

The space-time correlation of pressure in the measurement frame, $W_p(\vec{\zeta},\tau)$ can be found from

$$(4\text{-}52) \qquad W_p(\vec{\zeta},\tau) = \int_{-\infty}^{\infty} d\omega e^{-i\omega\tau}\ \hat{\theta}_p\ -$$

or be replacing ζ_1 by $\zeta_1 - u_o\tau$ in the rest-frame correlation

$W_p(\bar{\zeta},\tau)$ similarly computed from $\Theta_p(\bar{\zeta},\omega)$. From (4-50) and (4-51) we find

$$\hat{W}_p(\bar{\zeta},\tau) = C_w^2\rho^2\epsilon^{4/3}a^{-2/3}[(\zeta_1-u_o\tau)^2+\zeta_3^2+v_e^2\tau^2]^{1/3} \tag{4-53}$$

$$\times K_{2/3}(a[(\zeta_1-u_o\tau)^2+\zeta_3^2+v_e^2\tau^2]^{1/2}),$$

where $C_w^2 = 2^{-1/3}(16/9)[\Gamma(4/3)]^{-1}A^2$, in which the argument in square brackets may also be written as $u_e^2(\tau-\zeta_1/u_c)^2+\zeta_3^2+v_e^2\zeta_1^2$. In the inertial-subrange limit, where the argument of $K_{2/3}$ becomes small, we obtain for the decorrelation $\hat{\psi}_p(\bar{\zeta},\tau)$ $[= \hat{W}_p(0,0)-\hat{W}_p(\bar{\zeta},\tau)]$

$$\hat{\psi}_p(\bar{\zeta},\tau) = 2A^2\rho^2\epsilon^{4/3}[(\zeta_1-u_o\tau)^2+\zeta_3^2+v_e^2\tau^2]^{2/3} . \tag{4-54}$$

As apparent also from (4-33) and (4-24), the space-time isotropic approximation for the longitudinal decorrelation $\psi_1(r,\tau)$ in the inertial subrange and the quasinormality relation (4-25) thus lead to a space-time isotropic form (in two space variables) also for the (mean-rest-frame) pressure decorrelation. Since the result of the exact local-convection approximation to $\psi_1(r,\tau)$ was well approximated by space-time isotropy, as seen by (2-80) and Fig. 1, the exact approximation to $\hat{\psi}_p(\bar{\zeta},\tau)$ within the validity of the quasinormality assumption, is probably well approximated by (4-54). In the approximation of (4-27) and (4-28), by (4-54) we have

$$\psi_p(\bar{\zeta},\tau) = 2\rho^2\psi_1^2(\bar{\zeta},\tau);$$

this result, for $\tau = 0$, reduces to that obtained previously from the quasinormality assumption by Batchelor (1951) and Obukhov (1949).

We note the variously defined convection velocities for pressure. That defined by ζ_1/τ where $\partial\hat{W}_p/\partial\zeta_1 = 0$ is exactly u_o. According to approximation (4-53), that defined by ζ_1/τ where $\partial\hat{W}_p/\partial\tau = 0$ is u_c, given at (2-120), the same as the one defined at (4-50) by the cross-spectral density. The convection velocity defined by imposing Taylor's hypothesis (4-46), computed by use of (4 41), was u_e, given by (2-120). In short, these convection velocities referring to pressure are just the same as the corresponding ones referring to the streamwise velocity component in the same approximation of space-time isotropy of the longitudinal velocity correlation (see Table 2, last column).

We turn now to $P(\overline{K},\omega)$ in the low-wavenumber region, $K \ll \omega/v_o$, where we must abandon approximation (4-27) and consider dispersion. For this purpose we employ (4-26a). The integral (4-26a) derives contributions from domains where both, one, or neither of the factors E_4 has arguments corresponding to the local-convective peak in this function, as opposed to the dispersive tail; the peaks embrace roughly the ranges $|\omega'|/v_o k' \lesssim 1$ and $|\omega-\omega'|/v_o|\bar{k}-\bar{k}'| \lesssim 1$, as seen in Sec. 3.2.

We consider first the contribution to $P(\overline{K},\omega)$, say $P_c(\overline{K},\omega)$, from the non-dispersive domain where both E_4 factors are determined largely by local convection and are given approximately by Eq. (3-9). Examination of Eq. (4-26a) for $\omega L/v_o \gg 1$ yields in order of magnitude

(4-55) $$P_c(\overline{K},\omega) \sim \rho^2 \omega (\epsilon^{2/3} v_o^{-1} K^{-8/3})^2 (\omega/v_o K)^{-16/3}$$
$$= \rho^2 \epsilon^{4/3} v_o^{10/3} \omega^{-13/3}.$$

P_c thus has a form roughly independent of K and consistent with the

scaling (4-4). The domain of integration in (4-26a) that contributes to P_c has $k_2 \; \omega/v_o \sim (>>K)$, so that local convection in the normal direction can generate velocity fluctuations at frequency ω, though the local convection in the plane of $\overline{K}$ cannot. To be justified in regarding P_c as determined by the inertial subrange we must restrict consideration to the nonviscous regime where $\omega \ell_o/v_o << 1$, in accord with condition (2-93a).

On the basis of (4-55) we obtain for the contribution of P_c to the spectrum (4-10) of average pressure on a moving area of radius R_o where $\omega R_o/\pi v_o >> 1$ and $\omega R_o/\pi u_o >> 1$, the order of magnitude estimate

(4-56) $$Q_c(\omega) \sim \rho^2 \epsilon^{4/3} v_o^{10/3} \omega^{-13/3} R_o^{-2}.$$

The ratio of Q_c to the high-wavenumber mean-convective contribution $\hat{Q}_+$ was given at Eq. (4-16). It is noteworth that, on account of the sufficiency of condition (2-93a) for the inertial subrange in the local-convection approximation, it was not necessary to assume KL and L/R_o large to obtain the results (4-55) and (4-56).

We now attempt to estimate in order of magnitude the contributions to $P(\overline{K},\omega)$ from the domains where one or both E_4 factors are determined largely by dispersion. To do so we consider crudely that in the domain of the local-convective peak $E_4(k,\omega)$ is again given by (3-9) and outside that domain (but where condition (3-3) is still satisfied) by the dispersive form (3-11). Assuming $\omega L/v_o >> 1$, we join the domains at a value of $\omega/v_o k$ somewhat larger than unity. Examination of (4-26a) for $\omega/v_o K >> 1$ on this basis shows that the results depend essentially on the value of m in (3-4) and (3-11).

In particular, for $m = 3$ both the convective-dispersive and dispersive-dispersive contributions, defined with reference to the arguments of the E_4 factors in (4-26a), diverge on account of the dependence of $E_4(k',\omega')$, as given by (3-11), when $k'\to 0$ and $\omega'\to 0$. It is inferred that the actual values of these contributions, if $m = 3$, depend on $E_4(k',\omega')$ in the non-universal range where $k'L\lesssim 1$, $\omega'L/v_o\lesssim 1$ and the inertial subrange form (3-11) no longer applies even for $\omega'/v_o k'>>1$. In the range where conditions (2-93a) (left inequality) does not apply, we estimate the order of magnitude of $E_4(k,\omega)$, taking L^{-1} as the wavenumber scale and v_o/L as the frequency scale, as

(4-57) $$E_4(k,\omega)\sim v_o L^2,$$

or, if $kL<<1$, as $(kL)^4 v_o L^2$. We may then estimate the previously divergent convective-dispersive and dispersive-dispersive contributions to $P(\bar{K},\omega)$ with the integration domain properly restricted to the universal range (2-93a), and estimate the contribution from the non-universal domain separately by use of (4-57). We have contented ourselves with placing upper limits on the orders of these contributions. We find the non-universal contribution is at most of order $(\omega L/v_o)^{-4/3}$ relative to the universal convective contribution P_c of (4-55), and the convective-dispersive and dispersive-dispersive contributions are likewise of higher order relative to P_c with regard both to $(\omega L/v_o)^{-1}$ and to the other small expansion variable $(\omega/v_o K)^{-1}$. Hence, if $m = 3$ in (3-4) and the quasinormality form (3-26a) is roughly valid, at least, in the subject domains $P(\bar{K},\omega)$ is given approximately by the convective part P_c of (4-55) and $\hat{Q}_-$ by Q_c of (4-56), as stated in (4-12) and (4-14).

Assumption that $m = 1$ in (3-4) and (3-11) leads to different and rather curious results for the convective-dispersive and dispersive-dispersive contributions. Denoting these contributions to $P(\bar{K},\omega)$ respectively by P_{cd} and P_d, from (4-26a) once more for $\omega L/v_o >> 1$ we find these increase with decreasing K and estimate their orders of magnitude relative to P_c of (4-55) as given by

$$(4\text{-}58) \qquad P_{cd}/P_c \sim (\omega/v_o K)^{5/6}(KL)^{-5/6}, \quad P_d/P_{cd} \sim (KL)^{-5/6}.$$

Thus P_d is small relative to P_{cd} for $KL >> 1$, and in the non-dispersive limit where $KL \to \infty$ at fixed $\omega/v_o K$, P_{cd} in turn becomes small relative to P_c. In a domain where $\omega/v_o K \gtrsim KL$, however, P_{cd} exceeds P_c. As for the corresponding contributions to $\hat{Q}_-(\omega)$ of (4-13), where $\omega R_o/v_o >> 1 >> R_o/L$ and $\omega R_o/u_o >> 1$, the contribution Q_d from $P_d(\bar{K},\omega)$ is controlled by non-universal behavior at $K \lesssim L^{-1}$ but is estimated as $\sim (R_o/L)^{1/3}$, and hence small, relative to the contribution Q_{cd} from $P_{cd}(\bar{K},\omega)$. Relative to Q_c of (4-56) we then estimate $Q_{cd}/Q_c \sim (\omega R_o/v_o)^{5/6}(R_o/L)^{5/6}$; Q_{cd} varies as $R_o^{-1/3}$, rather than R_o^{-2}, and depends on v_o only via ϵ. Again, where $R_o/L \to 0$ the contribution Q_c from the K-independent P_c thus predominates, but where $\omega R_o/v_o \gtrsim L/R_o$ the largest contribution is Q_{cd}. The condition $L/R_o >> \omega R_o/v_o (>> 1)$ for applicability of (4-12) and (4-14) thus obtained for $m = 1$ is much more restrictive than the condition $\omega R_o/v_o >> 1$ obtained for $m = 3$. We have already given reason to suppose that $m = 3$ is the appropriate value.

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Thus P_d is small relative to P_{cd} for $KL>>1$, and in the non-dispersive limit where $KL\to\infty$ at fixed $\omega/v_o K$, P_{cd} in turn becomes small relative to P_c. In a domain where $\omega/v_o K \gtrsim KL$, however, P_{cd} exceeds P_c. As for the corresponding contributions to $\hat{Q}_-(\omega)$ of (4-13), where $\omega R_o/v_o >> 1 >> R_o/L$ and $\omega R_o/u_o >> 1$, the contribution Q_d from $P_d(\bar{K},\omega)$ is controlled by non-universal behavior at $K \lesssim L^{-1}$ but is estimated as $\sim(R_o/L)^{1/3}$, and hence small, relative to the contribution Q_{cd} from $P_{cd}(\bar{K},\omega)$. Relative to Q_c of (4-56) we then estimate $Q_{cd}/Q_c \sim (\omega R_o/v_o)^{5/6}(R_o/L)^{5/6}$; Q_{cd} varies as $R_o^{-1/3}$, rather than R_o^{-2}, and depends on v_o only via ε. Again, where $R_o/L\to 0$ the contribution Q_c from the K-independent P_c thus predominates, but where $\omega R_o/v_o \gtrsim L/R_o$ the largest contribution is Q_{cd}. The condition $L/R_o >> \omega R_o/v_o (>>1)$ for applicability of (4-12) and (4-14) thus obtained for $m = 1$ is much more restrictive than the condition $\omega R_o/v_o >> 1$ obtained for $m = 3$. We have already given reason to suppose that $m = 3$ is the appropriate value.

5 LOCAL-CONVECTION APPROXIMATION FOR SHEAR FLOW

It is suggested in further work to treat space-time correlations of velocity and pressure also in shear flow and, in particular, a turbulent boundary layer by extension of the present approach; the treatment, however, will become more a guide to construction of a heuristic model and less a systematic application of kinematic and similarity principles. We take one step here by suggesting the generalization to shear flow of the separation (2-6) of large-scale convective effects, and of the local-convection approximation (2-59).

We assume for the present a uniform turbulent shear flow, taking the flow in the x_1 direction and velocity gradient u' in the x_2 direction; relative to the mean rest frame for the fluid layer at $x_2 = 0$, the mean velocity is thus

(5-1) $$\bar{u}(x_2) = u' x_2 \bar{I}_1,$$

where $\bar{I}_1$ denotes a unit vector. We may denote the decorrelation tensor of the fluctuating velocity in this frame between space-time points $(\bar{x},t)$, $(\bar{x}+\bar{r},t+\tau)$ by $\psi_{ij}(x_2,x_2',\bar{\zeta},\tau)$, with $\bar{r} = (\zeta_1,x_2'-x_2,\zeta_3)$ and $\bar{\zeta} = (\zeta_1,\zeta_3)$, by virtue of homogeneity in the x_1-x_3 plane. Likewise, the decorrelation tensor measured in a frame in which the mean velocities at $\bar{x}$ and $\bar{x}'$ are equal but opposite, i.e., the frame with velocity $(1/2)[\bar{u}(x_2')+\bar{u}(x_2)]$ relative to the frame of definition of $\bar{u}(x_2)$, by the homogeneity in the x_2 direction is independent also of x_2 and x_2' except via $x_2'-x_2$ and may be denoted by $\psi^o_{ij}(\bar{r}',\tau)$, where $\bar{r}'$ denotes the

separation vector in the specified frame. By the usual kinematic transformation these correlations are related exactly by

$$(5\text{-}2) \qquad \psi_{ij}(x_2,x_2',\bar{r},\tau) = \psi^o_{ij}(\bar{r}-(1/2)[\bar{u}(x_2')+\bar{u}(x_2)]\tau,\tau) \ .$$

As in the instance of unsheared flow, we may expect that provided r and τ satisfy certain conditions, namely the previous conditions (2-5) with possibly another to be adjoined, we may regard the eddies that mainly determine $\psi^o_{ij}(\bar{r},\tau)$ as being statistically independent of the large eddies that contain most of the turbulence energy. Then, as at (2-6), we may approximately separate out the effect of the large eddies by defining an intrinsic decorrelation tensor $\tilde{\psi}_{ij}(\bar{r},\tau)$ relative to the locally co-moving frame which is independent of this motion:

$$(5\text{-}3) \qquad \tilde{\psi}^o_{ij}(\bar{r},\tau) = \int d^3\bar{v}P(\bar{v})\tilde{\psi}_{ij}(\bar{r}-\bar{v}\tau,\tau) \ .$$

$\tilde{\psi}_{ij}$ is still affected by the existence of shear. Now we distinguish a pseudo-convective effect of the shearing, just as we did of spatial fluctuating-velocity dispersion in the unsheared case in the discussion of Eq. (2-48) and in the appendix, Eqs. (A-5), (A-7). Specifically, we imagine the fluctuating velocities at the correlated points with their differing mean velocities as tending to be preserved as the medium between the points is distorted by shear. Thus, apart from residual dispersive effects, we expect that in the local proper frame in question the velocity correlation for the two space-time points will not change if, for any change $\delta\bar{r}'$ in

the separation vector $\bar{r}'$, a corresponding change $\delta\tau$ is made in the time separation τ such that points moving with the respective mean velocities and coinciding with $\bar{x}+\bar{r}'$, $\bar{x}$ respectively at times $t+\tau$, t would be separated by $\bar{r}'+\delta\bar{r}'$ at the respective times $t+\tau+\delta\tau$, $t+\delta\tau$. Hence, if we define a decorrelation tensor $\tilde{\psi}^c_{ij}$ with this pseudo-convective effect removed by writing

$$(5\text{-}4) \qquad \tilde{\psi}_{ij}(\bar{r}',\tau) = \tilde{\psi}^c_{ij}(\bar{r}'-[\bar{u}(x_2')-\bar{u}(x_2)]\tau,\tau) \ ,$$

then $\tilde{\psi}^c_{ij}(\bar{r}'',\tau)$ will depend on τ (at fixed $\bar{r}''$) only on account of the residual dispersive effect. Eqs. (5-3) and (5-4) yield

$$(5\text{-}5) \qquad \psi^o_{ij}(\bar{r},\tau) = \int d^3\bar{v}P(\bar{v})\tilde{\psi}^c_{ij}(\bar{r}-[\bar{v}+\bar{u}(x_2')-\bar{u}(x_2)]\tau,\tau)$$

and (5-2) then yields ψ_{ij} in terms of $\tilde{\psi}^c_{ij}$.

We define a non-dispersive approximation, generalizing the local-convection approximation for unsheared flow, by

$$(5\text{-}6) \qquad \tilde{\psi}^c_{ij}(\bar{r}'',\tau) \doteq \tilde{\psi}^c_{ij}(\bar{r}'',0) \ .$$

Since, by (5-5) without approximation, $\psi^o_{ij}(\bar{r},0) = \tilde{\psi}^c_{ij}(\bar{r},0)$, Eqs. (5-5) and (5-6) yield

$$(5\text{-}7) \qquad \tilde{\psi}^o_{ij}(\bar{r},\tau) \doteq \int d^3\bar{v}P(\bar{v})\tilde{\psi}^o_{ij}(\bar{r}-[\bar{v}+\bar{u}(x_2')-\bar{u}(x_2)]\tau,0)$$

as the expression of the space-time correlation in terms of the space correlation in the non-dispersive approximation. In the frame referred to by (5-2) this similarly becomes

$$(5\text{-}8) \qquad \psi_{ij}(x_2,x_2',\bar{\zeta},\tau) \doteq$$

$$\int d^3\bar{v}P(\bar{v})\psi_{ij}(x_2,x_2',\bar{\zeta}-[(3/2)\bar{u}(x_2')-(1/2)\bar{u}(x_2)+\bar{v}]\tau,0).$$

If the shear is truly uniform, as assumed at (5-1), the square-bracketed effective local convection velocity in (5-8) may be written also as $(1/2)[\bar{u}(x_2') + \bar{u}(\bar{x}_2)] + u' I_1(x_2' - x_2) + \bar{v}$. The consequences of this non-dispersive approximation can be extended to the treatment of pressure also in the present anisotropic situation by assumption again of a quasinormal velocity distribution. (The mean shear-turbulence interaction contribution naturally must now be included.)

As for the conditions on $\bar{r}$ and τ defining the universal range where (5-3) may be applied, we apparently require, apart from the conditions (2-5), that $[u(x_2') - u(x_2)]\tau \ll L$; since, however, in shear flows typically $v_o \sim u'L$, the added condition is normally already implied by conditions (2-5).

As written, Eq. (5-7) or (5-8) is meaningful also when applied to a non-uniform shear flow, e.g., a boundary layer. If it is so applied, the crudity of the approximation, at best, must be greater or the domain of applicability smaller. In any case, it remains true, as we saw in the corresponding instance in unsheared flow, that the non-dispersive approximation does not suffice to give a good approximation to the tail of the wavenumber-frequency spectrum where $\omega/v_o k$ is large.

Other well known complications are involved in treating pressure fluctuations in a boundary layer, as opposed to the unsheared homogeneous flow considered in the preceding sections, even with regard to the domain of the convective peak in the wavenumber spectrum to which we might tentatively apply

(5-8). These effects are attended by the entrance of the length parameter ν/v_* characterizing the viscous-sublayer thickness and, with reference to spatially dependent quantities, also the distance x_2 from the wall*.

*Among other consequences of the wall, the spectrum $\hat{P}(\bar{K},\omega)$ of pressure on the wall in the approximation of incompressibility, streamwise homogeneity, and vanishing viscosity, varies as K^2 when $K \ll \delta^{-1}$, where δ is the boundary-layer thickness, and hence vanishes as $K \to 0$, in contrast to the nonvanishing limit $\hat{P}(0,\omega)$ in the unsheared homogeneous case (Kraichnan 1956, Phillips 1956).

6. SUMMARY AND CONCLUSION

Kolmogorov's principles, as expressed by Eq. (2-6), provide a basis for the treatment of Eulerian space-time correlations in the universal range (2-5) in homogeneous turbulence by permitting the explicit separation of the kinematic effect of convection by the large eddies. From the related presumption that the space-time correlation in the local co-moving frame has the commonly accepted similarity character independent of the velocity distribution $P(\bar{v})$, and from the order of magnitude of the coefficients involved, it was indicated that the local-convection (non-dispersive) approximation (2-59) has validity in the inertial and viscous subranges.

In the local-convection approximation, with an isotropic, normal velocity distribution, the space-time structure functions of velocity were computed explicitly for the inertial subrange [Eq. (2-80) and Fig. 1] and for the viscous subrange [Eq. (2-86)]. These functions were found to be nearly space-time isotropic in the inertial subrange and exactly so in the viscous, with differing velocity scales for longitudinal and transverse components. The result in the former case was compared with the space-time correlation in grid turbulence measured by Favre and associates. The computed envelope of the time correlations at fixed spatial separations, for the presumed value of turbulence intensity in the experiment, decreases somewhat too slowly to agree well with the measured

one. This discrepancy is presumably due mainly to neglect of dispersion, which is significant at the Reynolds number of the experiment. At higher Reynolds number, such fitting of the space-time correlation may usefully supplement measurements of pure spatial correlations and wavenumber spectra to yield the Kolmogorov coefficient [Eq. (2-56)].

For the unsheared homogeneous turbulence assumed in this work, Taylor's hypothesis relating wavenumber spectra in the mean rest frame to frequency spectra in the measurement frame (velocity $-\bar{u}_o$) was examined for the inertial subrange. In the local-convection approximation the hypothesis was found to be exact for arbitrary turbulence intensity v_o/u_o, provided an appropriate effective convection velocity is used to relate wavenumber to frequency for the respective spectra [Eq. (2-116)]. For small v_o/u_o, this convection velocity differs from u_o by a term of relative order v_o^2/u_o^2 depending also on the quantity whose spectra are in question, and other commonly defined convection velocities also differ by such terms [Table 2]. The related cross-spectral density of streamwise fluctuating velocity in the measurement frame was also computed in the space-time isotropic approximation [Eq. (2-119)].

The local-convection approximation for fixed finite scale L of the energy-containing eddies does not yield the wave-number-frequency spectrum of turbulence correctly up to arbitrarily large values of $\omega/v_o k$; rather, velocity dispersion in space

and time due to local eddy distortion enters, and it is necessary to consider the frequency spread of the intrinsic (proper-frame) spectrum $\tilde{E}_4(k,\omega)$. The standard similarity dependence of $\tilde{E}_4$ for the inertial subrange was accepted [Eq. (3-1)], and an argument given for the dependence $\tilde{E}_4(k,\omega)\propto k^4$ as $k^2\epsilon/\omega^3 \to 0$. At small $v_o k/\omega$, the intrinsic and mean-rest-frame spectra, $\tilde{E}_4$ and E_4, could then be inferred uniquely within a constant factor [Eqs. (3-11), (3-15) with m = 3]. An explicit form having plausible properties was suggested to be used for $\tilde{E}_4(k,\omega)$ [Eq. (3-4)]. Its most consequential property is to have a finite second moment with respect to frequency [Eq. (A-18), n = 2]. The resulting dispersive correction to the rest-frame spectrum as given by the local-convection approximation was computed for moderate $\omega/v_o k$ [Eq. (3-10)]. At given k, dispersion increases the energy $E_4(k,\omega)$ in the higher range of $\omega/v_o k$ at the expense of that in the lower.

From the assumed intrinsic spectrum, the dispersive departure from validity of Taylor's hypothesis was found. The departure from unity of the pertinent ratio of spectra at wavenumber k_1 is of the order of $(v_o/u_o)^2(k_1L)^{-2/3}$ and hence, for $k_1L \gg 1$, small relative to the non-dispersive departure compensable by the above mentioned redefinition of convection velocity [Eqs. (3-21), (3-24)]. The dispersive correction to the local-convection approximation to the space-time velocity correlation was also found in two limits; it is of the relative order of $(v_o\tau/L)^{2/3}$ if $(v_o\tau/L)^{1/2} \ll 1$ and $(v_o\tau)^{3/2}/rL^{1/2} \gg 1$, and the order of $(v_o\tau)^2/r^{4/3}L^{2/3}$ if $v_o\tau/r \ll 1$ [Eqs. (3-28) - (3-31)]. The effect on the comparison with measured results for grid turbulence was computed. The magnitude of this effect depends on the ratio

B/A_o [Eqs. (2-44), (2-45)] measuring the relative decorrelating influence of time delay and of spatial separation in the proper frame; for a likely magnitude of B/A_o and the Reynolds number of the experiment the computed dispersive effect is substantial.

Inferences from kinematic and similarity arguments for an inertial subrange can be drawn also concerning pressure spectra in the measurement frame. The wavenumber-frequency spectrum $\hat{P}(\bar{K},\omega)$ referring to a plane parallel to $\bar{u}_o$ was indicated to have the similarity form (4-5) in the neighborhood of its convective ridge ($|\omega-\bar{u}_o\cdot\bar{K}| \lesssim v_oK$). For $v_o/u_o<<1, \omega L/u_o>>1$, and $\omega \ell_o/v_o<<1$, the point frequency spectrum $\hat{P}(\omega)$ could be inferred from (4-5) to have the dependence (4-8), varying as $\omega^{-7/3}$. The spectrum $\hat{Q}(\omega)$ of average pressure for a circular area of radius R_o, though not directly measurable, was considered with a view to future treatment of a turbulent boundary layer. For a small area ($\omega R_o/u_o<<1$), the area correction is given by (4-11). For a large area ($\omega R_o/u_o \gtrsim 2\pi$), contributions associated with the convective peak of $P(\bar{K},\omega)$ and with the low-wavenumber region ($K \lesssim 2\pi R_o^{-1}$) are distinguished, the former varying as R_o^{-3}. From the form of $P(\bar{K},\omega)$ in the convective region the former contribution is given by (4-15).

With assumption of quasinormality of the velocity distribution and the non-dispersive approximation of space-time isotropy for the longitudinal fluctuating velocity correlation, the spectrum $\hat{P}(\bar{K},\omega)$ applicable in the neighborhood of the convective ridge was computed explicitly for the inertial subrange [Eq. (4-40), a = 0] (and formally extended into the non-universal range, assumed isotropic). In this approximation the point spectrum $\hat{P}(\omega)$ was computed explicitly for arbitrary v_o/u_o

[Eq. (4-45)], and the cross-spectral density and space-time correlation of pressure were also obtained [Eqs. (4-50), (4-53), (4-54)]. The variously defined convection velocities are the same as those for the streamwise velocity component in the same space-time isotropic approximation. From the assumption of quasinormality, $\hat{P}(\bar{K},\omega)$ was estimated also in the low-wavenumber part of the inertial domain where $\omega/v_oK \gg 1$, $L^{-1} \ll \omega/v_o \ll \ell_o^{-1}$, and $\omega/u_oK \gg 1$. Throughout this domain (assuming m = 3 in (3-11)), $P(\bar{K},\omega)$ is given approximately by the local-convective wavenumber-independent form (4-55). Correspondingly, the low-wavenumber contribution to the moving-area spectrum $\hat{Q}(\omega)$ for $\omega R_o/v_o \gg 1$ and $\omega R_o/u_o \gg 1$ assumes the form (4-56), varying as R_o^{-2}. The ratio of the low-wavenumber to the mean-convective contribution to $\hat{Q}(\omega)$ is given by (4-16).

The kinematic separation of large-scale convective effects basic to this work, and the related local-convection approximation for space-time correlations, were plausibly extended to the more general case of shear turbulent flow [Eqs. (5-5), (5-8)].

The present work provides a basis for further comparisons with measurements on grid-produced and other relatively unsheared turbulent flows and a springboard for an attack on boundary-layer turbulence with reference to properties not yet adequately explored.

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APPENDIX

PROPERTIES OF THE INTRINSIC SPACE-TIME CORRELATION OF VELOCITY IN THE INERTIAL SUBRANGE

If we propose to calculate explicit corrections to the local-convection approximation (2-54)[or (2-59)] in the inertial subrange, i.e., to consider terms of next lowest order in R_c/L (with R_c given by (2-52) and L the scale of the energy-containing eddies), we should examine further the degree of validity of the basic separation (2-6) of large-scale convection. We therefore consider the possibility that the intrinsic decorrelation $\tilde{\psi}(\bar{r},\tau)$ for given $\bar{v}$ depends significantly on $\bar{v}$ and likewise on the scale L, the rms velocity fluctuation v_0, and any parameters of large-scale anisotropy. We denote such possible dependence in a partially suppressed fashion by writing $\tilde{\psi} = \tilde{\psi}_{\bar{v}}(L^{-1},\bar{r},\tau)$. Later we must specify more precisely the definition of $\bar{v}$. Eq. (2-6) is thus generalized to

$$\psi(\bar{r},\tau) = \int d^3\bar{v}P(\bar{v})\tilde{\psi}_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,\tau). \tag{A-1}$$

Now let $\tilde{\psi}(\bar{r},\tau)$ again denote a suitable function of the form $\tilde{\psi}(\bar{r},\tau) = (\epsilon r)^{2/3}F(\epsilon^{1/3}r^{-2/3}|\tau|)$[Eq. (2-42)] independent of $\bar{v}$ and the large eddies; $\tilde{\psi}(\bar{r},\tau)$ is to be so chosen, if possible, that when inserted in place of $\tilde{\psi}_{\bar{v}}(\bar{r},\tau)$ in (A-1), the computed correction $\Delta\psi(\bar{r},\tau) \equiv \psi(\bar{r},\tau)-\psi_c(\bar{r},\tau)$ to the local-convection approximation

$$\psi_c(\bar{r},\tau) = \int d^3\bar{v}P(\bar{v})\psi(\bar{r}-\bar{v}\tau,0) \tag{A-2}$$

is correct to lowest order in R_c/L. For any $\tilde{\psi}(\bar{r},\tau)$ we can

rewrite (A-1) as an (exact) equation for $\Delta\psi$:

$$\text{(A-3)} \qquad \Delta\psi(\bar{r},\tau) = \int d^3\bar{v}P(\bar{v})[\tilde{\psi}_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,0)-\psi(\bar{r}-\bar{v}\tau,0)]$$

$$+\int d^3\bar{v}P(\bar{v})[\tilde{\psi}_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,\tau)-\tilde{\psi}_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,0)] \ .$$

The first integral represents an increment due to non-vanishing L^{-1} at zero dispersive time delay, and the second represents an increment due to the dispersive time delay at fixed $\bar{v}$ and L. The objective is then to define $\bar{v}$ and choose $\tilde{\psi}(\bar{r},\tau)$ such that $\Delta\psi$ in (A-3), to lowest order in R_c/L, is equal to

$$\text{(A-4)} \qquad \int d^3\bar{v}P(\bar{v})[\tilde{\psi}(\bar{r}-\bar{v}\tau,\tau)-\psi(\bar{r}-\bar{v}\tau,0)].$$

With regard to the first integral in (A-3), by (A-1) we have $\int d^3\bar{v}P(\bar{v})\tilde{\psi}_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,0) = \psi(\bar{r},0)$; furthermore, to the lowest two orders in r/L, $\psi(\bar{r},0)$ has the form

$$\psi(\bar{r},0)\rightarrow F(0)(\epsilon r)^{2/3}-A_L v_o^2(r/L)^2, \qquad (A_L = \text{constant})$$

as exemplified by the result for the von Karman interpolation form (Hinze 1959, Eq. (3-136)). Hence, the contribution to $\Delta\psi$ from the first integral in (A-3), relative to the magnitude of ψ_c, is of the order $-v_o^2(R_c/L)^2\div(\epsilon R_c)^{2/3}\sim-(R_c/L)^{4/3}$. On the other hand, the value of (A-4) on the basis of a $\tilde{\psi}(\bar{r},\tau)$ characterized by (3-5) is estimated by reference to Eqs. (3-28), (3-29), to lowest order in L^{-1}, as of relative order $(R_c/L)^{2/3}$ and hence of lower order than the first term in (A-3). It remains then to consider whether the second term in (A-3) alone agrees with (A-4) in this lowest order.

We are free to choose $\bar{v}$ in (A-1)(and hence (A-3) and (A-4)) to include contributions to fluctuating velocity from all eddies down to some minimum size ~L', say (cf. Inoue 1951); for example, we might set $L' \approx R_c(L/R_c)^{\delta}$ for some $0 < \delta < 1$ before letting $R_c/L \to 0$. The smaller L' is, the less these eddies are removed from the size $\sim R_c$ of eddies mainly determining $\psi_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,\tau)$ (with $|\bar{r}-\bar{v}\tau|\sim R_c$), and hence the more likely $\psi_{\bar{v}}$ is to depend significantly on $\bar{v}$ on account of distortion of these eddies by eddies of size $\geq$ L'. On the other hand, unless $L'\sim R_c$(i.e. $\delta = 0$), then $\psi_{\bar{v}}(L^{-1},\bar{r}',\tau)$ will have to include, by means of its dependence on the dispersive time delay τ, the decorrelating effect during τ due to the residual convection velocity arising from eddies in the size interval from ~L' down to $\sim R_c$. Consideration on this basis indicates, indeed, that the second term in (A-3) is of lower order in R_c/L than (A-4), and hence cannot reduce to the latter unless $L'\sim R_c$. If $L'\sim R_c$, the desired reduction can occur if the intrinsic decorrelation $\tilde{\psi}_{\bar{v}}(L^{-1},\bar{r}-\bar{v}\tau,\tau)$ (with $|\bar{r}-\bar{v}\tau|\sim R_c$) is statistically independent of the motion due to eddies of size $\sim R_c(L/R_c)^{\delta}$ even for $\delta \to 0$. We leave further clarification to explicit dynamic treatments (e.g., Kraichnan 1959, 1966) and, as stated in the text, accept the similarity form (2-42) for use in (A-1) where a definite form is required.

As a guide to the possible properties of the intrinsic decorrelation $\tilde{\psi}(r,\tau)\,[\equiv \tilde{\psi}(\bar{r},\tau)]$, assumed of the form (2-42), we append discussion of certain conjectures. Previously we distinguished two sources of decorrelation in $\tilde{\psi}(r,\tau)$ relative to

$\tilde{\psi}(r,0)\,[=\psi(r,0)]$: (1) differential convection due to velocity dispersion over the distance r acting during time τ; (2) pseudo-Lagrangian velocity wander during τ, existing even when $r = 0$ With regard to the latter, since $\tilde{\psi}(0,\tau)\sim\epsilon|\tau|$, this effect may be considered a random walk of the velocity at a given point in the proper frame, corresponding in the related Lagrangian case to white noise in acceleration*.

For identifiable effects contributing to velocity decorrelation with time in a given Eulerian frame, one approximate mode of inclusion to be considered for possible validity is this: the effect is regarded as producing during τ a displacement vector, with some independent probability density, and for each realization of this displacement the velocity correlation is supposed to be formed as the Eulerian correlation function that applies in the absence of the effect with a displacement argument that is the vector sum of the previously applicable one with the new one. This mode is exemplified by the basic Eq. (2-6) that treats the effect of large-eddy motion on time correlation in the mean rest frame. A second simple mode of inclusion of a given decorrelating effect may be considered: the effect is regarded as producing a statistically independent contribution, not to the displacement vector in the correlation function, but to the velocity decorrelation itself. The former approach is surely indicated (through statistical independence may be doubtful) where the effect in question is

*Though it is useful to distinguish these effects, we cannot expect that the actual dynamics permit a clean-cut separation of the two.

due to an identifiable convection of a presumed relatively static eddy configuration, either a convection common to both spatial points in the frame of the Eulerian correlation in question or (in cruder approximation) one that arises from a relative velocity between the two points*.

Of the two effects distinguished in the first paragraph, the first, due to velocity dispersion over r, should thus be treated by the convection view (recall also the discussion of Eq. (5-4)). Which, if either, treatment is appropriate for the second effect, velocity wander, is less clear. The view that velocity wander contributes to an effective displacement vector is akin to the independence hypothesis of Corrsin 1961, Saffman 1963, referring to the relation of Lagrangian and Eulerian correlation functions. If we support this view, we incline to a rough conjectured form

$$\text{(A-5)} \quad \tilde{\psi}(r,\tau) = A_o \epsilon^{2/3} \int d^3\bar{x}_1 \int d^3\bar{x}_2 P(\bar{x}_1,\bar{x}_2) \left|\bar{r} - c_1 v_r \tau \bar{x}_1 - c_2 v_\tau \tau \bar{x}_2\right|^{2/3},$$

where $v_\tau = (\epsilon|\tau|)^{1/2}$ and v_r was defined after (2-43). $P(\bar{x}_1,\bar{x}_2)$ is the joint probability of contributions to the effective displacement vector, for points separated by $\bar{r}$, of $c_1 v_r \tau \bar{x}_1$ due to differential convection and $c_2 v_\tau \tau \bar{x}_2$ due to velocity wander; the probability variables $\bar{x}_1, \bar{x}_2$ are assumed normalized, so that the rms values of the two displacement contributions are $c_1 v_r|\tau|$ and $c_2 v_\tau|\tau|$, and c_1, c_2 are of the order of unity since v_r is of the order of the differential velocity over r

*We note that the assumption of Chandrasekhar (1956) corresponded to the opposite view that even the decorrelation with time due to local convection by large eddies appears in the form of an independent additive component in the velocity decorrelation [see Eq. (2-50)].

and v_τ of the order of the velocity wander during τ. In the spirit of the conjectured approximation, we regard $P(\bar{x}_1,\bar{x}_2)$ in (A-5) as independent of $\bar{r}$, τ and of the directions of $\bar{x}_1$, $\bar{x}_2$, except that it may depend significantly on μ_1 where $\mu_i \equiv \bar{x}_i \cdot \bar{r}/|\bar{x}_i| r$, i.e., the probability of a relative velocity between points $(\bar{x},t)$ and $(\bar{x}+\bar{r},t+\tau)$ may well depend on the direction with respect to the separation vector $\bar{r}$. If in addition, the processes associated with $\bar{x}_1$ and $\bar{x}_2$ may be considered independent, we may set

$$\text{(A-6)} \qquad P(\bar{x}_1,\bar{x}_2) = P_1(\bar{x}_1)P_2(\bar{x}_2) = (4\pi)^{-2}p_1(x_1,\mu_1)p_2(x_2).$$

If we support rather the view that velocity wander contributes directly and independently to the velocity decorrelation, we write in place of (A-5)

$$\text{(A-7)} \qquad \tilde{\psi}(r,\tau) = A_o\epsilon^{2/3} \int d^3\bar{x}_1 P_1(\bar{x}_1)\left|\bar{r}-c_1 v_r \tau \bar{x}_1\right|^{2/3} + (\epsilon r)^{2/3}G(z),$$

where z was defined by (2-43). The second term represents velocity wander; by (2-45) $G(z) \propto z$ as $z \to \infty$, but $G(z)$ may vanish rapidly enough as $z \to 0$ so that the first term alone gives $\tilde{\psi}(r,\tau)-\tilde{\psi}(r,0)$ to lowest order in z. The simplest assumption for G(z), however, would be simply $G(z) = Bz$ whence

$$\text{(A-8)} \qquad (\epsilon r)^{2/3}G(z) = \tilde{\psi}(0,\tau) = B\epsilon|\tau| .$$

This form, being independent of r, yields a contribution to $\tilde{E}(k,\tau)$ or $\tilde{E}_4(k,\omega)$ having singular dependence as $\delta(k)$ and hence could be useful, if at all, only for quantities entailing integration over wave number.

If we could suppose that $P_1(\bar{x}_1)$ in (A-7) is isotropic (i.e., independent of μ_1) and also normal, we obtain, as at Eq. (2-80) (i=0), a contribution to $\tilde{\psi}(r,\tau)$ from the first term in (A-7) given by

$$A_o(\epsilon\tilde{R})^{2/3}[1-H_o(c_1 z)]$$

where

$$\tilde{R}^2 = r^2+s_o^2(v_r\tau)^2 = r^2(1+s_o^2 z^2) \ .$$

An analysis of the corresponding contribution to the spectral density $\tilde{E}_4(k,\omega)$ appears to yield a negative value (varying as $\epsilon^{5/2}k^2\omega^{-13/2}$) in the limit $\Omega \equiv \omega/\epsilon^{1/3}k^{2/3} \to \infty$, however, an unacceptable result.

Similarly, examination of the $\tilde{E}_4(k,\omega)$ given by (A-5) in the simplified case of $c_1 = 0$ with isotropic $P_2(\bar{x}_2)$ yields $\tilde{E}_4(k,\omega) \propto - E(k)\epsilon k^2\omega^{-4} (\propto -\epsilon^{5/3}k^{1/3}\omega^{-4})$ as $\Omega\to\infty$, i.e., a negative spectrum again in this limit. It is natural to consider also the variant of (A-5), in the simplified case where $c_1 = 0$, obtained by adding displacements r and $c_1 v_r \tau$ in quadrature:

(A-9) $$\tilde{\psi}(r,\tau) = A_o\epsilon^{2/3}(r^2+c_2^2\epsilon|\tau|^3)^{1/3}.$$

This form also yields

(A-10) $$\tilde{E}_4(k,\omega) \propto -\epsilon^{5/3}k^{1/3}\omega^{-4} \quad \text{as} \quad \Omega\to\infty \ ;$$

in fact, by virtue of the discontinuous third derivative with respect to τ, any form leading to

$$\tilde{E}(k,\tau) \to A(k) - B(k)|\tau|^3 \quad \text{as } \tau\to 0 \ ,$$

where the function $B(k) > 0$ for some k, yields this same negative

result for such k in the limit $\Omega \to \infty$.

A similar variant on the first term of (A-7) (or on (A-5) with $c_2 = 0$), namely

(A-11) $$\tilde{\psi}(r,\tau) = A_0 \epsilon^{2/3}(r^2 + c_1^2 \epsilon^{2/3} r^{2/3} \tau^2)^{1/3} ,$$

on the other hand, yields a positive result*

(A-12) $$\tilde{E}_4(k,\omega) \propto \epsilon^{5/2} k^2 \omega^{-13/2} \quad \text{as } \Omega \to \infty .$$

The quadrature variant of the complete form (A-5) is

(A-13) $$\tilde{\psi}(r,\tau) = A_0 \epsilon^{2/3}(r^2 + c_1^2 \epsilon^{2/3} r^{2/3} \tau^2 + c_2^2 \epsilon |\tau|^3)^{1/3} .$$

It appears that, despite the c_1 term, the term $\propto |\tau|^3$ will cause this form to yield once more a negative $\tilde{E}_4(k,\omega)$ in the limit of large Ω, so that this form also fails in this regime. Though (A-13) is thus unacceptable even as a tentative approximation, it may correctly indicate the dependence of $\tilde{\psi}(r,\tau)$ in the limits of large and small z.

In the limit $z \to 0$, if $\int d^3 x_1 P(\bar{x}_1,\bar{x}_2)\mu_1 = 0$, form (A-5) yields to lowest order in z

(A-14) $$\tilde{\psi}(r,\tau) - \tilde{\psi}(r,0) \propto \tilde{\psi}(r,0) z^2 \propto \epsilon^{4/3} r^{-2/3} \tau^2 .$$

In particular, if (A-6) holds and

(A-15) $$\int_0^\infty dx x^3 \int_{-1}^{1} d\mu \, \mu p_1(x,\mu) = 0 ,$$

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*More generally, an assumed form

$$\tilde{\psi}(r,\tau) = A_0 \epsilon^{2/3}(r^{1/m} + c^2 \epsilon^{2/3} r^{1/m - 4/3} \tau^2)^{2m/3}$$

(which has the required form $(\epsilon r)^{2/3} F(z)$) also yields the result (A-12). If $m = 1/2$, this form reduces to (A-11); if $m = 3/4$, it has instead the required behavior, ($\propto |\tau|$) as $z \to \infty$.

then (A-14) is the limiting form. But Eq. (A-15) must be valid, in fact, since otherwise the separation of all pairs of fluid elements, on the average, would be increasing, contrary to the condition of compressibility. Likewise, form (A-7), if $\int d^3\bar{x}_1 P_1(\bar{x}_1)\mu_1 = 0$ and $G(z) \to 0$ faster than z^2 as $z \to 0$, also yields this result; so also does form (A-13) (cf. Eq. (2-48)). On the other hand, if $G(z) \propto z$ as $z \to 0$, which would be so if form (A-8) is valid in this limit, form (A-7) yields instead

(A-16) $$\tilde{\psi}(r,\tau)-\tilde{\psi}(r,0) \to \tilde{\psi}(0,\tau) \propto \epsilon|\tau|;$$

in such case, velocity wander rather than differential convection represents the leading contribution.

In the opposite limit where $z \to \infty$, if (A-6) holds, form (A-5) yields to lowest order

$$\tilde{\psi}(r,\tau)-\tilde{\psi}(0,\tau) \propto \psi(0,\tau)z^{-1} \propto (\epsilon r)^{2/3};$$

form (A-13) also yields this result. On the other hand, form (A-7) in this limit yields

$$\tilde{\psi}(r,\tau)-\tilde{\psi}(0,\tau) \propto (\epsilon r)^{2/3}z^{2/3} = \epsilon^{8/9}r^{2/9}|\tau|^{2/3}.$$

We can avoid constructing functions $\tilde{\psi}(r,\tau)$ that lead to negative $\tilde{E}_4(k,\omega)$ by starting from the latter. In the inertial subrange (3-3) the requisite form for $\tilde{E}_4(k,\omega)$ is given by (3-1) in the text. Form (3-1) yields

(A-17) $$\tilde{\psi}(r,0) = (9/10)\Gamma(1/3)N_0(\epsilon r)^{2/3}, \quad \tilde{\psi}(0,\tau) = (3/2)\pi N_1\epsilon|\tau|,$$

where

(A-18) $$N_n = 2\int_0^\infty d\Omega\,\Omega^n G(\Omega),$$

i.e., the similarity form (3-1) implies the requisite limiting

forms (2-44) and (2-45) independently of $G(\Omega)$, provided only that the moments N_0 and N_1 exist and do not vanish.

Expressions (A-17) represent the leading terms of the general similarity form $\mathbb{T}(r,\tau) = (\varepsilon r)^{2/3}F(z)$ in the respective limits $z\to 0$ and $z\to\infty$, and we may consider the z-dependence of the next-to-leading terms in these limits. The dependence as $z\to 0$ is determined by the behavior of $G(\Omega)$ as $\Omega\to\infty$. First, suppose $G(\Omega)$ in this limit decreases faster than Ω^{-3}. From Eqs. (2-15) and (3-1), by expansion of $1-\cos\omega\tau$, we readily find

(A-19) $$F(z)\to(9/10)\Gamma(1/3)N_0+(3/2)\Gamma(2/3)N_2z^2 \text{ as } z\to 0,$$

the first term having been given at (A-17). Now suppose instead

$$G(\Omega)\to g_\infty\Omega^{-(2+\delta)} \text{ as } \Omega\to\infty,\ 0<\delta<1,\ g_\infty \text{ constant.}$$

In this instance we find

(A-20) $$F(z)\to(9/10)\Gamma(1/3)N_0+b_0g_\infty z^{1+\delta} \quad \text{as } z\to 0,$$

$$b_0 \equiv \frac{2\pi}{1-2\delta/3}\,\frac{\Gamma(2\delta/3)\cos(\pi\delta/3)}{\Gamma(2+\delta)\cos(\pi\delta/2)}\ .$$

Convergence or non-convergence of the second moment N_2 of $G(\Omega)$, which led respectively to (A-19) or (A-20), is equivalent to decrease of $\tilde{\Theta}(r,\omega)$, the frequency transform of the proper-frame spatial correlation, more or less rapidly than ω^{-3} as $\varepsilon^{-1/3}r^{2/3}\omega\to\infty$.*

---

*We may consider also a hypothetical space-independent velocity-wander term (A-8) in $\mathbb{T}(r,\tau)$, which implies a next-to-leading term $\propto z$ in $F(z)$ as $z\to 0$, in place of those of (A-19) or (A-20). The corresponding contribution to $\tilde{E}_4(k,\omega)$ would be $\pi^{-1}B\varepsilon\delta(k)\omega^{-2}$; this singular function is the limit of one of the form (3-1), formally corresponding to $G(\Omega) = \pi^{-1}B\Omega^{-7/2}\delta(\Omega^{-3/2})$. Such an $\tilde{E}_4(k,\omega)$ may be regarded as the limit of that given by form (3-4) with $m = 1$ as $\gamma_m\to 0$, in the sense that $\pi^{-1}x/(k^2+x^2)\to\delta(k)$ as $x\to 0$.

Similarly, the dependence of $F(z)$ as $z\to\infty$ is related to the behavior of $G(\Omega)$ as $\Omega\to 0$. In particular, if

$$G(\Omega)\to g_o\Omega^{-1+\Delta} \quad \text{as } \Omega\to 0,\ 0<\Delta<1,\ g_o \text{ constant},$$

we find

$$\text{(A-21)} \qquad F(z)\to(3/2)\pi N_1 z+2\pi\frac{-\Gamma(-(5+2\Delta)/3)}{\Gamma(1-\Delta)}\,\frac{\cos[(4+\Delta)\pi/3]}{\cos[(1-\Delta)\pi/3]}\,g_o z^{-\Delta}$$

as $z\to\infty$, the first term having been given at (A-17). The condition $\Delta>0$ is required for finiteness of N_o in (A-17). If $\Delta = 1/2$, this result may be written

$$\text{(A-22)} \qquad F(z)\to(3/2)\pi N_1 z + \pi(\pi/2)^{1/2}\,g_o z^{-1/2} \quad \text{as } z\to\infty.$$

The type of function assumed for $G(\Omega)$ at Eq. (3-4) meets the conditions yielding the limiting forms for $F(z)$ in (A-19) and (A-22) above, and Eqs. (3-5) and (3-6) accordingly represent particular cases of these results. Apart from the coefficient α_m , form (3-5) is thus correct provided only that the second moment N_2 of $G(\Omega)$ is actually finite. Further, the form (A-19) (and (3-5)) agree with the limiting form suggested by (A-5) and (A-13), or, more generally, with the limiting form obtained if differential convection determines $\tilde{\psi}(r,\tau)-\tilde{\psi}(r,0)$ as $z\to 0$. At the same time, in the opposite limit where $z\to\infty$, if differential convection contributed to effective displacement in quadrature with the contribution from velocity wander, as in (A-5) and (A-13), the next-to-leading term in $F(z)$ would be constant ($\Delta\to 0$ in (A-21)), not $\propto z^{-1/2}$ as in (A-22).

(A-18) $\tilde{E}_4(k,\omega) = \epsilon^{1/3}k^{-7/3}G(\Omega), \qquad \Omega \equiv \epsilon^{-1/3}k^{-2/3}|\omega|,$

in the domain (3-3). Form (A-17) is the limit of such a form, formally corresponding to $G(\Omega) = \pi^{-1}B\Omega^{-7/2}\delta(\Omega^{-3/2})$. Form (A-18) implies

(A-19) $\tilde{\psi}(r,0) = (9/10)\Gamma(1/3)N_o(\epsilon r)^{2/3}, \quad \tilde{\psi}(0,\tau) = (3/2)\pi N_1\epsilon|\tau|,$

where

(A-20) $$N_n = 2\int_0^\infty d\Omega\Omega^n G(\Omega),$$

i.e., the similarity form implies the requisite limiting forms (2-44) and (2-45) independently of $G(\Omega)$, provided only that the moments N_o and N_1 exist and do not vanish. It is natural to examine the consequences of replacing $\delta(k)$ in (A-17) by a common function, $\pi^{-1}\beta/(k^2+\beta^2)$, which reduces to it in the limit $\beta \to 0$, but introduces a finite scale of spatial correlation of appropriate order ($\sim\epsilon^{1/2}\omega^{-3/2}$). In view of (A-18) we therefore assume

(A-21) $\tilde{E}_4(k,\omega) = \pi^{-2}B\epsilon\omega^{-2}\cdot\gamma\epsilon^{-1/2}|\omega|^{3/2}/(k^2+\gamma^2\epsilon^{-1}|\omega|^3),$

(A-22) i.e., $G(\Omega) = \pi^{-2}B\gamma\Omega^{-1/2}(1+\gamma^2\Omega^3)^{-1}.$

The limit (A-17) would correspond to $\gamma \to 0$. Eq. (A-21) further yields, by use of (2-15), the limiting forms of $\tilde{\psi}(r,\tau)$ given at (3-5) and (3-6). Hence, the dependence of the two leading terms of $\tilde{\psi}(r,\tau)$ as given by (A-21) in both limits $z \to 0$ and $z \to \infty$ agrees with the limiting forms suggested by (A-5) and (A-13), or more generally with the limiting forms obtained if differential convection determines $\tilde{\psi}(r,\tau)-\tilde{\psi}(r,0)$ as $z \to 0$ and,

as $z \to \infty$, contributes to effective displacement in quadrature with the contribution from velocity wander.

We may inquire how general is the dependence $F(z)-F(0) \propto z^2$ in the limit $z \to 0$, implied by the specific form (A-21) as stated at (3-5), ie., the dependence $\Delta_\tau \tilde{\psi} \equiv \tilde{\psi}(r,\tau)-\psi(r,0) \propto \tau^2$ as $\epsilon^{1/3} r^{-2/3} \tau \to 0$. The singular form (A-17), for example, corresponds rather to $\Delta_\tau \tilde{\psi} \propto |\tau|$. In general, we have by the inverse of the proper-frame analog of (2-26)

$$\Delta_\tau \tilde{\psi} = 2\int_0^\infty d\omega (1-\cos\omega\tau) \tilde{\theta}(r,\omega).$$

Hence, provided only that the frequency transform of the spatial correlation trace falls off more rapidly than ω^{-3} for all $r \neq 0$, i.e., that $G(\Omega)$ falls off more rapidly than Ω^{-3}, we have in fact $\Delta_\tau \tilde{\psi} \propto \tau^2$ as $\epsilon^{1/3} r^{-2/3} \tau \to 0$, since $\Delta_\tau \tilde{\psi}$ then approaches the result obtained from the leading term $1-\cos\omega\tau \to (1/2)(\omega\tau)^2$; explicitly,

$$\Delta_\tau \tilde{\psi} \to (3/2)\Gamma(2/3) N_2 (\epsilon r)^{2/3} (\epsilon^{1/3} r^{-2/3} \tau)^2$$

in terms of the assumed convergent moment N_2 defined by (A-20). On the other hand, if $\tilde{\theta}(r,\omega)$ falls off as $\omega^{-(2+\delta)}$, $0 < \delta < 1$, i.e.,

$$G(\Omega) \to g_o \Omega^{-(2+\delta)} \quad \text{as } \Omega \to \infty \quad (g_o \text{ constant}),$$

we obtain

$$\Delta_\tau \tilde{\psi} \to b_o g_o (\epsilon r)^{2/3} (\epsilon^{1/3} r^{-2/3} |\tau|)^{1+\delta},$$

$$b_o \equiv \frac{2\pi}{1-2\delta/3} \frac{\Gamma(2\delta/3)\cos(\pi\delta/3)}{\Gamma(2+\delta)\cos(\pi\delta/2)}$$

It seems likely that the second moment N_2 of $G(\Omega)$ is finite through we cannot exclude the contrary. If it is, then assumption of (A-21) can do no worse than yield numerical constants, e.g., C_o in (3-5), whose values may differ somewhat from the correct ones.

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FIGURE 1. Functions $H_i(v_o\sqrt{\tau}/r)$ of Eq. (2-80). These functions express the fractional departure from space-time isotropy of the velocity decorrelation functions computed in the local-convection approximation for the inertial subrange [(1) longitudinal, (2) transverse, (0) trace].

FIGURE 2. Space-time correlations for streamwise fluctuating velocity in laboratory frame. Experimental curves — — — — —, as given for grid turbulence by Favre (1965), Figs. 1,5; computed curves for inertial subrange by local convection approximation, Eq. (2-90): ————————, turbulence intensity $v_o/u_o = 0.027$, spatial separations aligned with mean flow ($\varphi = 0$); ——— — ———, $v_o/u_o = 0.027$ but misalignment $\varphi = 1.5°$, or $\varphi = 0$ but intensity arbitrarily adjusted to $v_o/u_o = 0.042$. Computed curves for vanishing separation coincide. Results are omitted at correlations < 0.2.

FIGURE 3. Space-time correlations for streamwise fluctuating velocity in laboratory frame. Experimental curves — — — — —, as in Fig. 2; computed curves for inertial subrange with dispersion: ——— ——— —, envelope from non-dispersive computation (————) of Fig. 2; — — — ———, $v_o/u_o = 0.027$, $\varphi = 0$, dispersion included by Eqs. (3-30), (3-31) with $\alpha_m = 2.31$; ——————, $v_o/u_o = 0.027$, $\varphi = 0$, dispersion included by space-independent addition to decorrelations with $B/A_o = 1.04$.

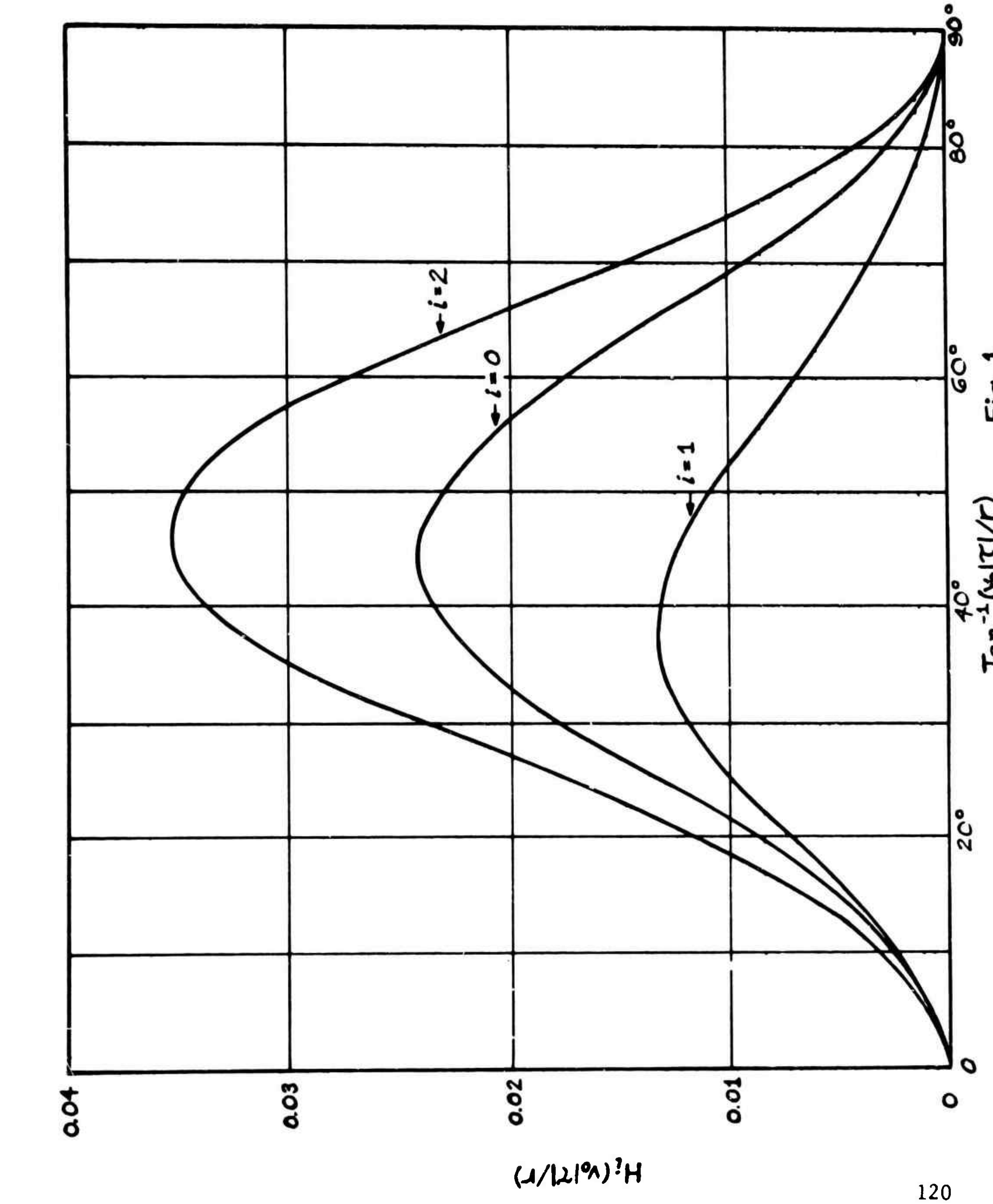

Fig. 1

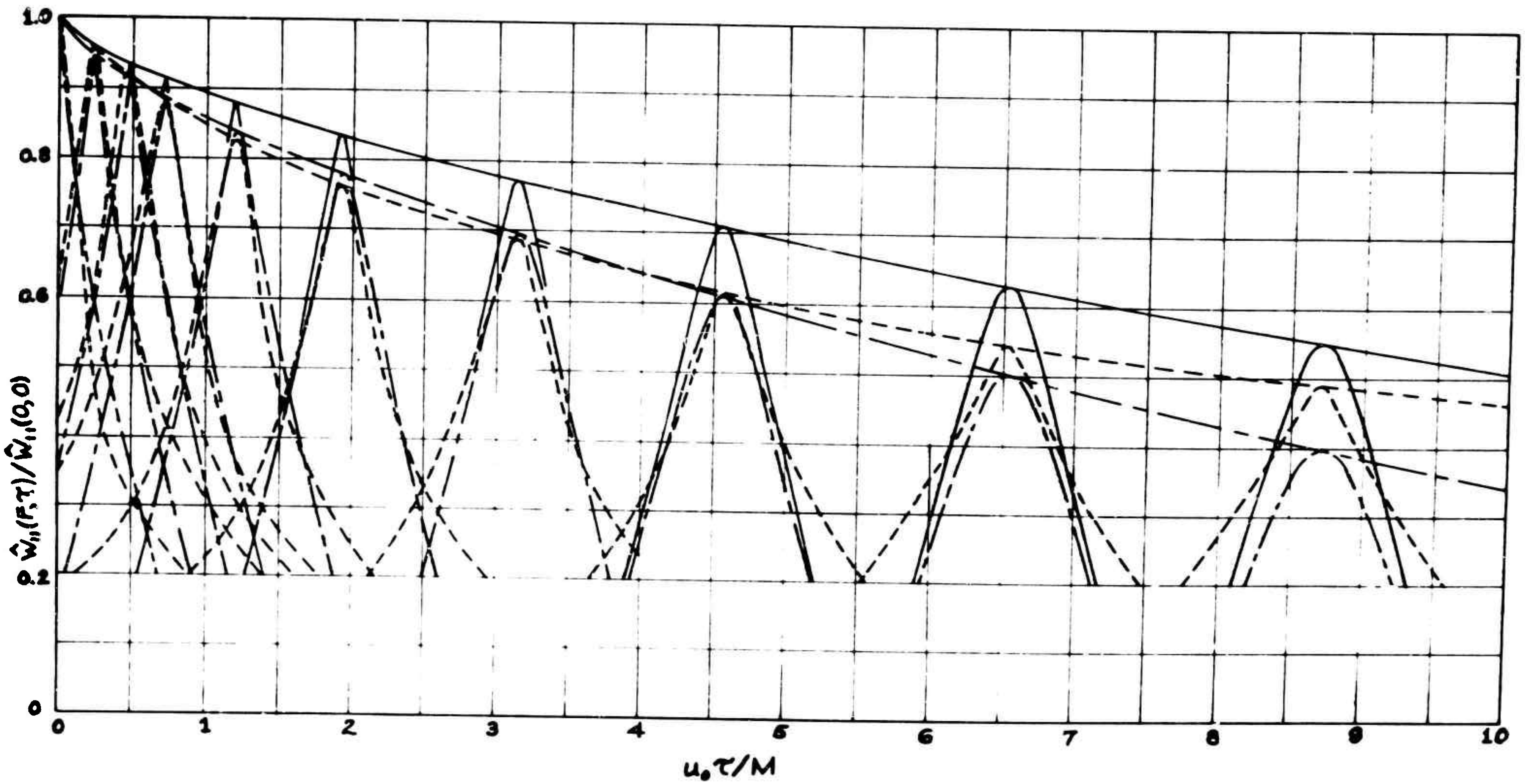

Fig. 2

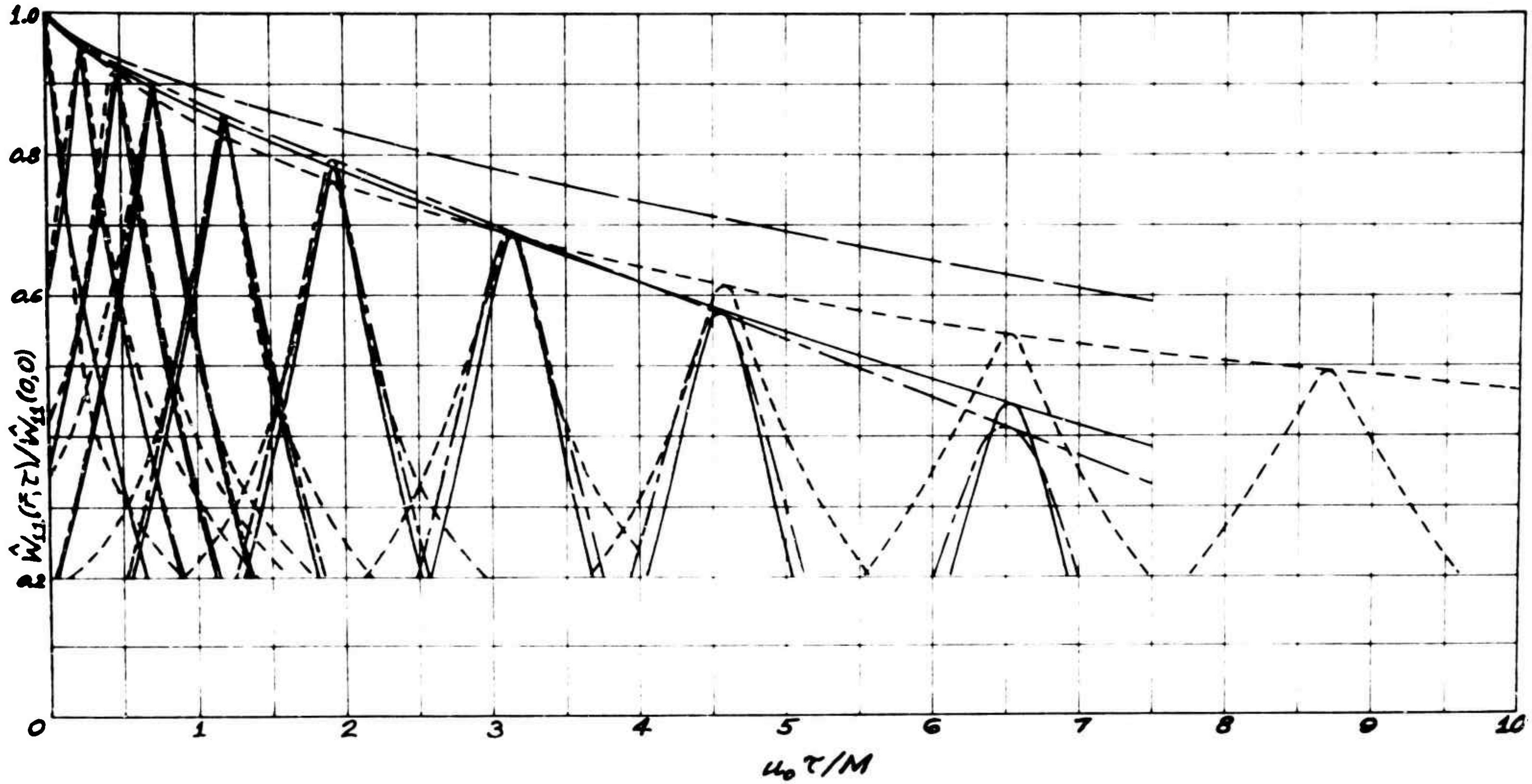

Fig. 3

local-convection approximation for the inertial subrange, to be exactly valid for all v_o/u_o provided a particular effective convection velocity different from u_o is assumed; for sufficiently large k_1L, where L is the scale of energy-containing eddies, the dispersive correction to this result is negligible. A plausible explicit form is proposed for the intrinsic energy spectrum in the co-moving frame, and consequent corrections to results of the local-convection approximation for space-time correlations and spectra are computed. This dispersive correction to correlations is appreciable at Reynolds numbers of typical grid-turbulence experiments. An extension of the basic separation of convection and the local-convection approximation to shear flow is suggested.

From kinematic and similarity arguments for the inertial subrange, inferences are also made concerning pressure spectra in a measurement frame with velocity $(-u_o,0,0)$ relative to the unsheared flow. A similarity form results for the wavenumber-frequency spectrum $\hat{P}(\overline{K},\omega)$ $[\overline{K} = (k_1, k_3)]$ in the vicinity of the convective ridge $(|\omega - k_1u_o| \lesssim v_oK)$. For $v_o/u_o << 1$, the functional form is then obtained for the point frequency spectrum $\hat{P}(\omega)(\propto\omega^{-7/3})$; the convective contribution to the spectrum $\hat{Q}(\omega)$ of average pressure on a moving circular area of radius R_o for $\omega R_o/u_o >> 1$ is smaller by a factor $0.8(\omega R_o/u_o)^{-3}$. On assumption of quasinormality of the velocity distribution and use of the non-dispersive approximation of space-time isotropy for the longitudinal correlation, the spectrum $\hat{P}(\overline{K},\omega)$ in the vicinity of the convective ridge is determined explicitly, as well as the cross-spectral density and space-time correlation of pressure. From quasinormality, $\hat{P}(\overline{K},\omega)$ is estimated also in the low-wavenumber domain. In the limit $\omega R_o/v_o >> 1$ the ratio of the low-wavenumber to the mean-convective contribution to the moving-area spectrum $\hat{Q}(\omega)$ is $\sim(v_o/u_o)^{10/3}(\omega R_o/u_o)$.

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13 ABSTRACT

Kolmogorov's principles provide a basis for the treatment of Eulerian space-time correlations for turbulence in the universal range by explicit separation of the kinematic effect of convection by large eddies. With reference to unsheared homogeneous turbulence, the usual similarity forms are assumed for intrinsic velocity correlations in a local co-moving frame. The local-convection approximation, neglecting dispersion in this frame and hence relating space-time correlations to pure spatial correlations, is indicated to be a useful one in the inertial and viscous subranges. For an isotropic normal velocity distribution, the structure functions of fluctuating velocity are computed and found to be nearly space-time isotropic in the former subrange and exactly so in the latter. The wavenumber-frequency spectrum of energy in the mean rest frame, $E_4(k,\omega)$, however, in the regime of large $\omega/v_o k$, where v_o denotes rms velocity, is not given correctly by the local-convection approximation, but essentially involves dispersion. Taylor's hypothesis relating wavenumber (k_1) spectra in the mean rest frame to frequency spectra in a measurement frame with velocity $-\bar{u}_o$ is found, in the

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Errata

for Report TRG-011-TN-67-1, "SPACE-TIME CORRELATIONS OF VELOCITY AND PRESSURE AND THE ROLE OF CONVECTION FOR HOMOGENEOUS TURBULENCE IN THE UNIVERSAL RANGE," by David M. Chase, April, 1967

p. 15, line preceding Eq. (2-14): For "energy of" read "energy by."

p. 18, line (-8): For "$P(\bar{v})v^2dv$" read "$P(v)v^2dv$."

p. 21: In Eq. (2-38), in the definitions of ξ_1 and ξ_2, and in the line following, for "v" read "ν."

p. 25, second line before Eq. (2-52): For "$\bar{v}$" read "v."

p. 30, (2-66): For "$v^{1/2}$" read "$\nu^{1/2}$."

p. 37, line (-9): For "relation (2-64)" read "relation (2-59)."

p. 45, line 6: Insert comma after "u_o."

p. 45, Eq. (2-100): For "θ" read "$\hat{\theta}$."

p. 46, Eqs. (2-104), (2-105): For "$P(v)$" read "$P(\bar{v})$."

p. 48, line 2: For "for" read "For."

p. 48, lines 9 and 11: For "r" read "$\bar{r}$" and for "r_1" read "$\bar{r}_1$" in arguments.

p. 49, line (-3): For "θ_{ii}" read "$\hat{\theta}_{ii}$."

p. 49, line 8: For "u'_{ij}" read "u_{ij}."

p. 50, Eq. (2-119): Insert (after [in second line.

p. 56, line 12: Close parentheses after "respectively."

p. 60, line (-5): Close parentheses after "(3-6)."

p. 61, line (-5): Close parentheses in "$\Gamma(1/3)$."

p. 62, line (-10): For "γ" read "γ_m."

p. 63, Eq. (3-18), and p. 64, Eq. (3-20): Move factor "$k^{2(m-1)}$" to follow "$\int_o^\infty dk$"; for "γ^2" read "γ_m^2."

p. 63, Eq. (3-18): For "θ" read "$\hat{\theta}$."

p. 64, Eq. (3-21): For "$b_m(B/A_o)^2$" read "$b_o \alpha_m$."

p. 64, line (-4): For "b_1 and b_3 were" read "b_o was."

p. 65, line 6: For "ΔT" read "ΔT_c."

p. 68, line (-7): For "β_o" read "$\hat{\beta}_o$."

p. 72, Eq. (4-5): For "$\bar{u}_o \cdot K$" read "$\bar{u}_o \cdot \bar{K}$."

p. 73, line 6: Read "$\hat{P}(\omega) \propto \omega^{-7/3}$."

p. 80, line 1: For "k << 1" read "$k \ll L^{-1}$."

p. 80, line 3: For "$P(K,\omega)$" read "$P(\bar{K},\omega)$."

p. 84, line 12: For "K_1" read "k_1."

p. 86, line 7: Delete $\pm$ before k_{1m}.

p. 87, Eq. (4-52): Should read "$\hat{W}_p(\bar{\zeta},\tau) = \int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\, \hat{\theta}_p(\bar{\zeta},\omega)$."

p. 90, line 2: Read "$k_2 \sim \omega/v_o (\gg K)$."

p. 94: Insert comma after "$\tilde{\psi}_{ij}(\bar{r},\tau)$," line 10, and after "frame," line 11.

p. 106, line (-2): For "through" read "though."

pp. 114-116: <u>Delete these pages</u>.

p. 117, seventh reference: Following "Bolt" add "Beranek & Newman Rept. No. 1310."